

Efficiency

Announcements

Measuring Efficiency

Recursive Computation of the Fibonacci Sequence

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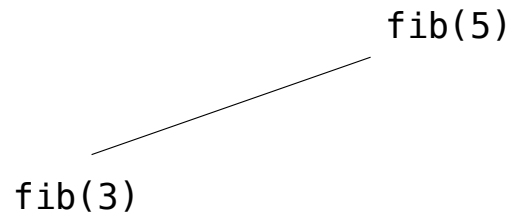
fib(5)

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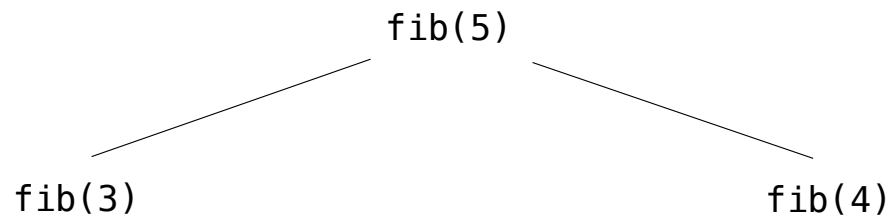


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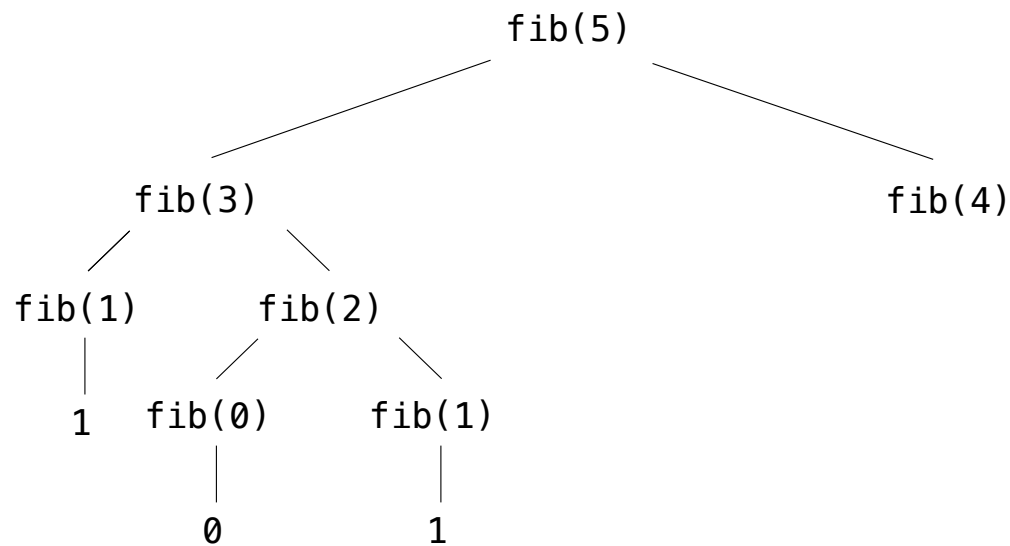


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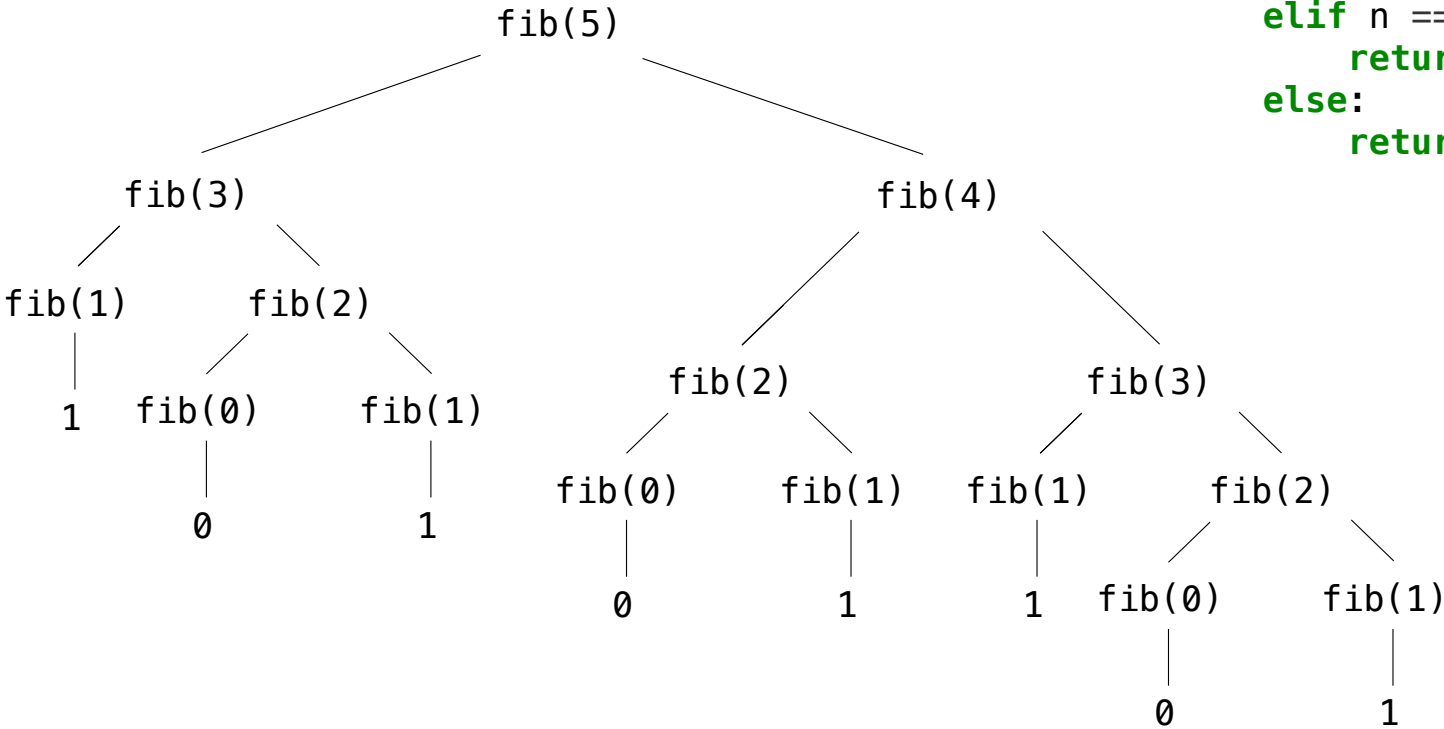


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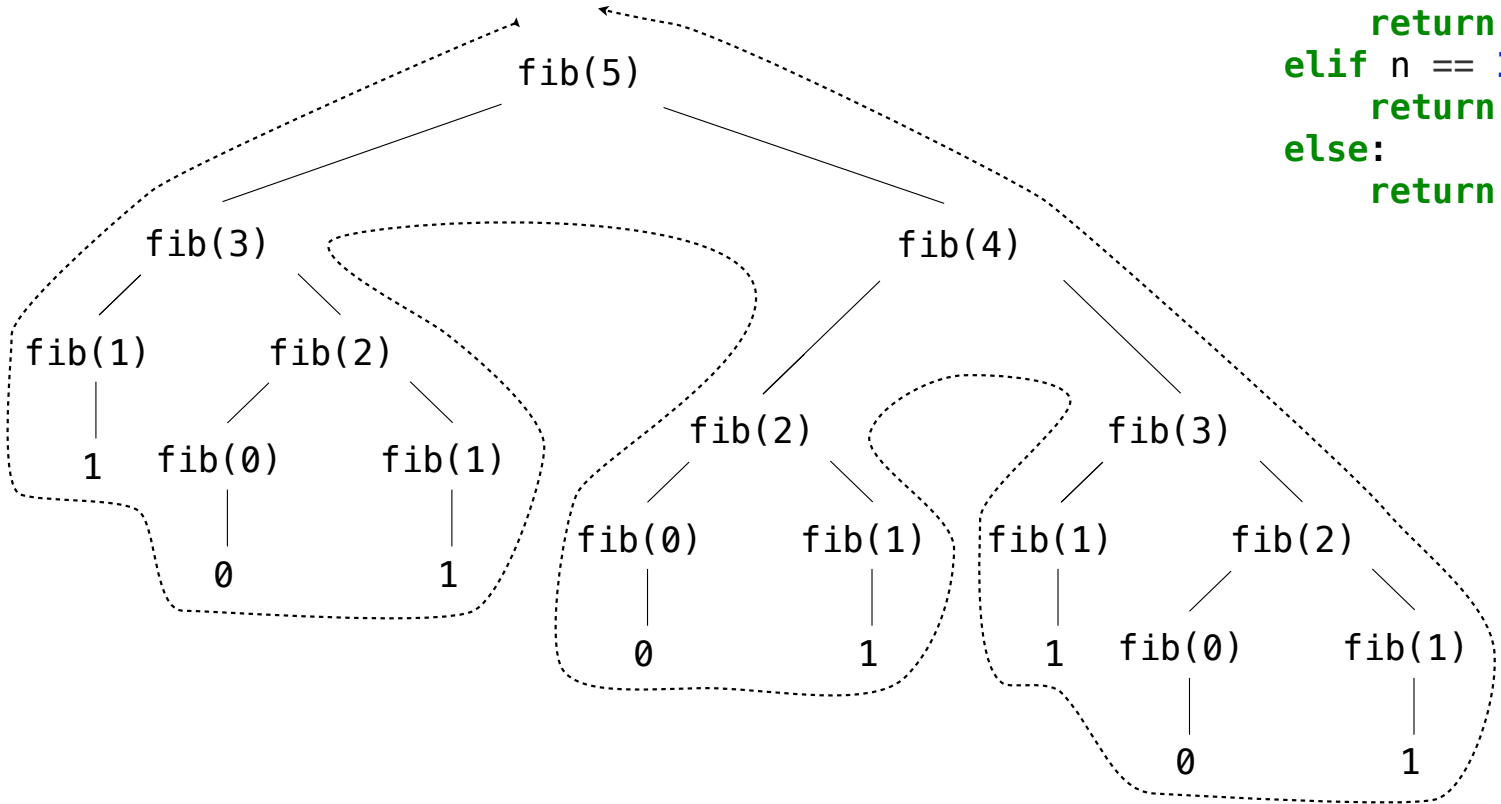


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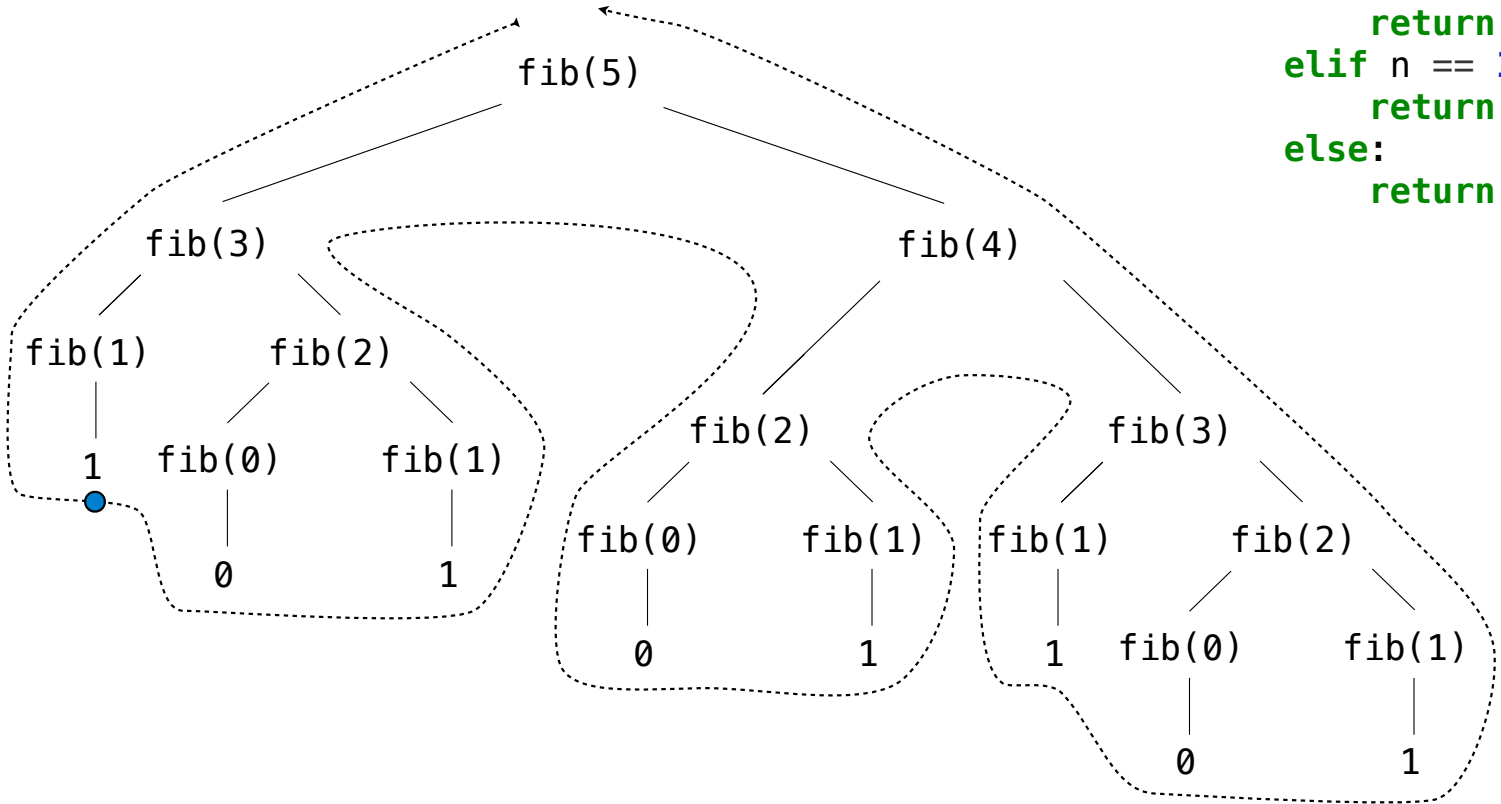


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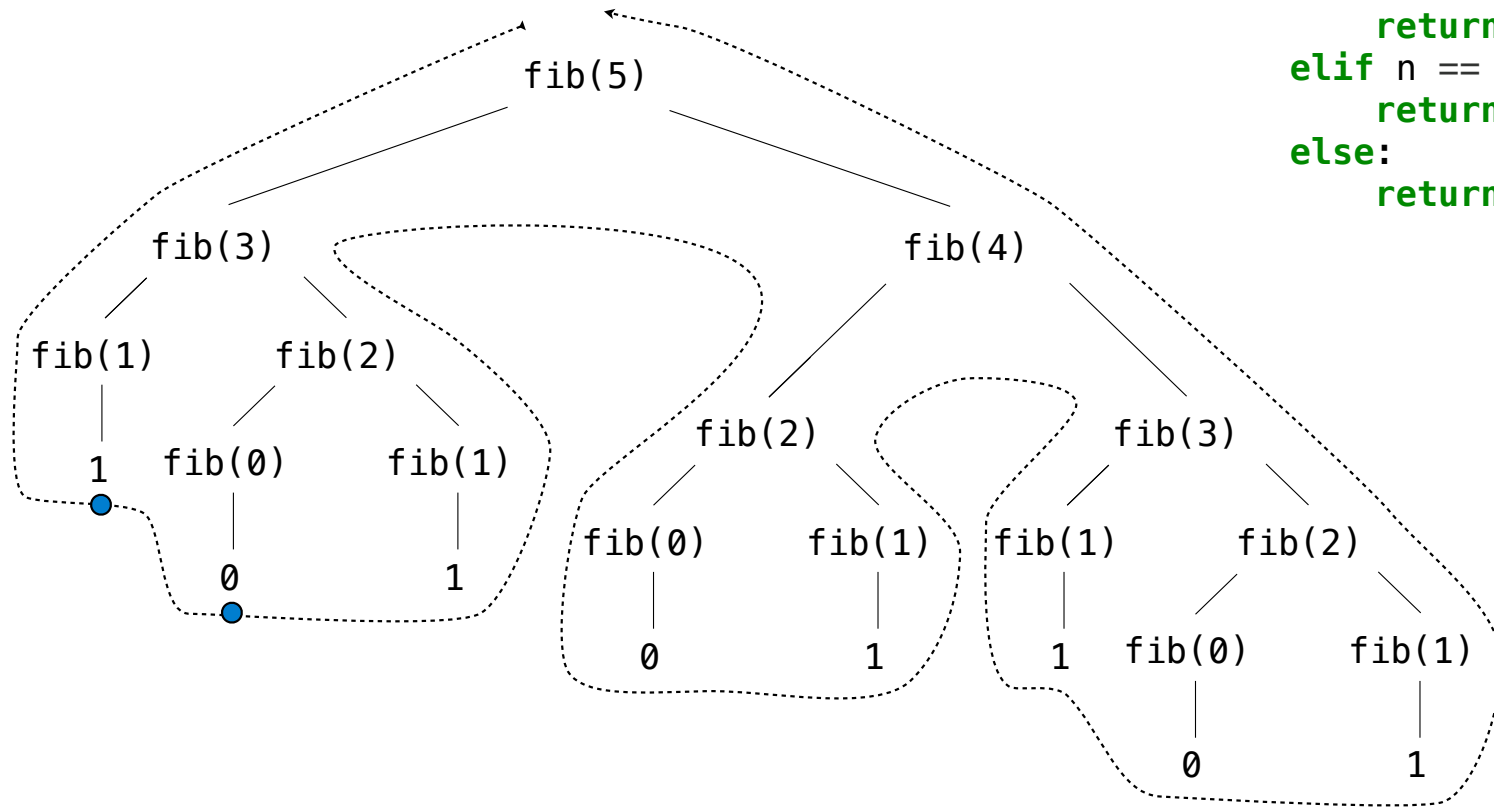


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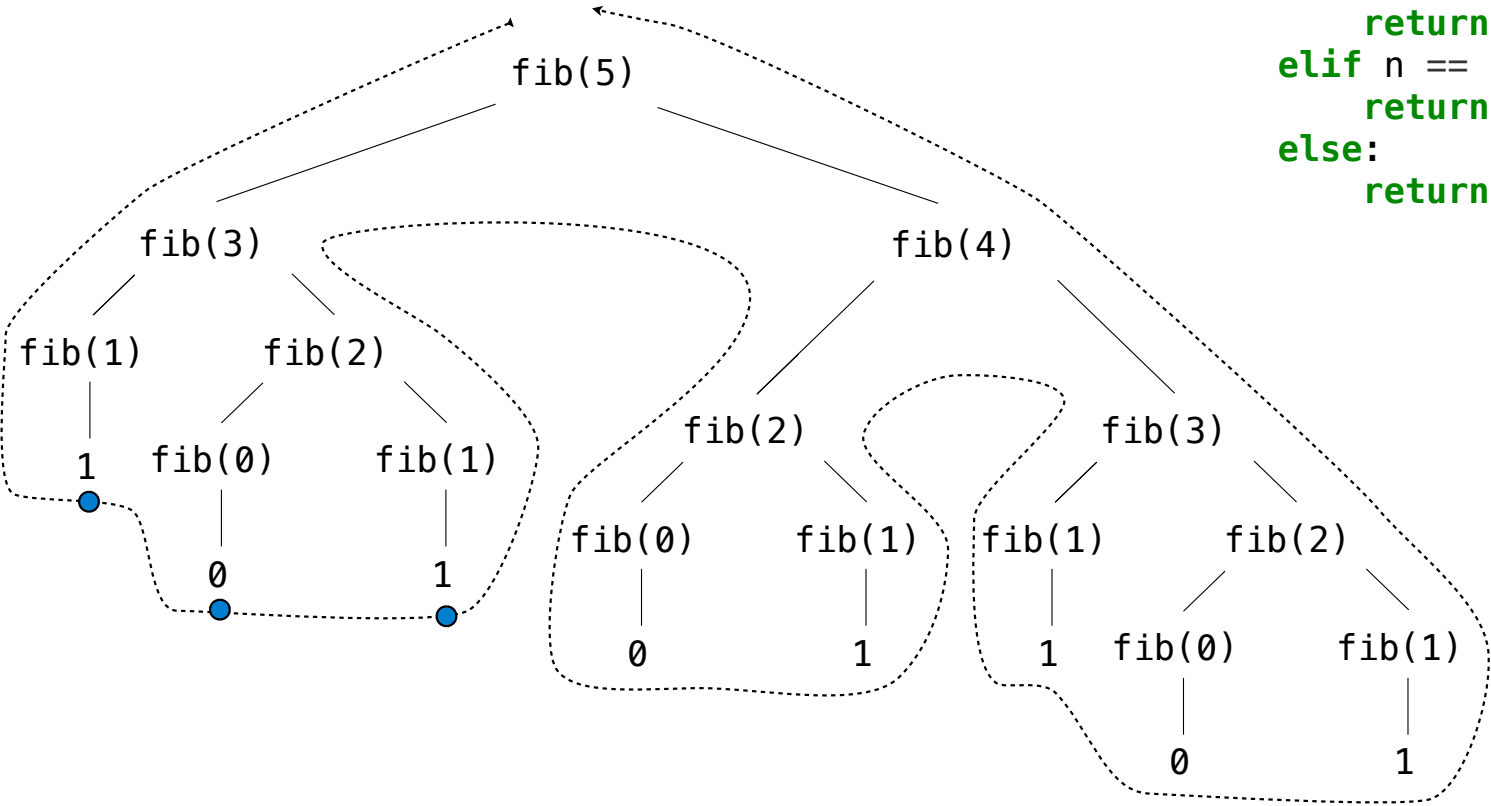
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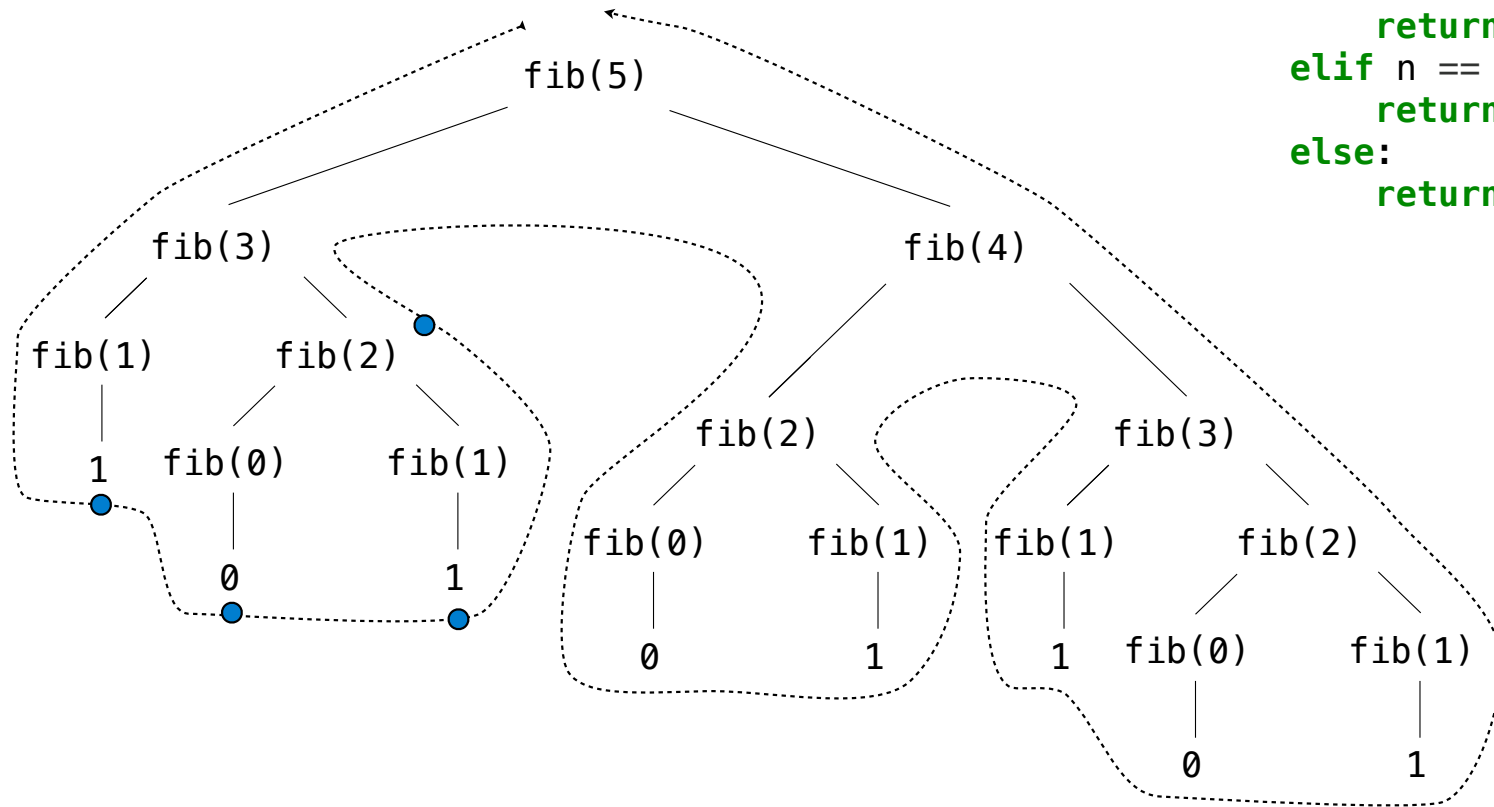
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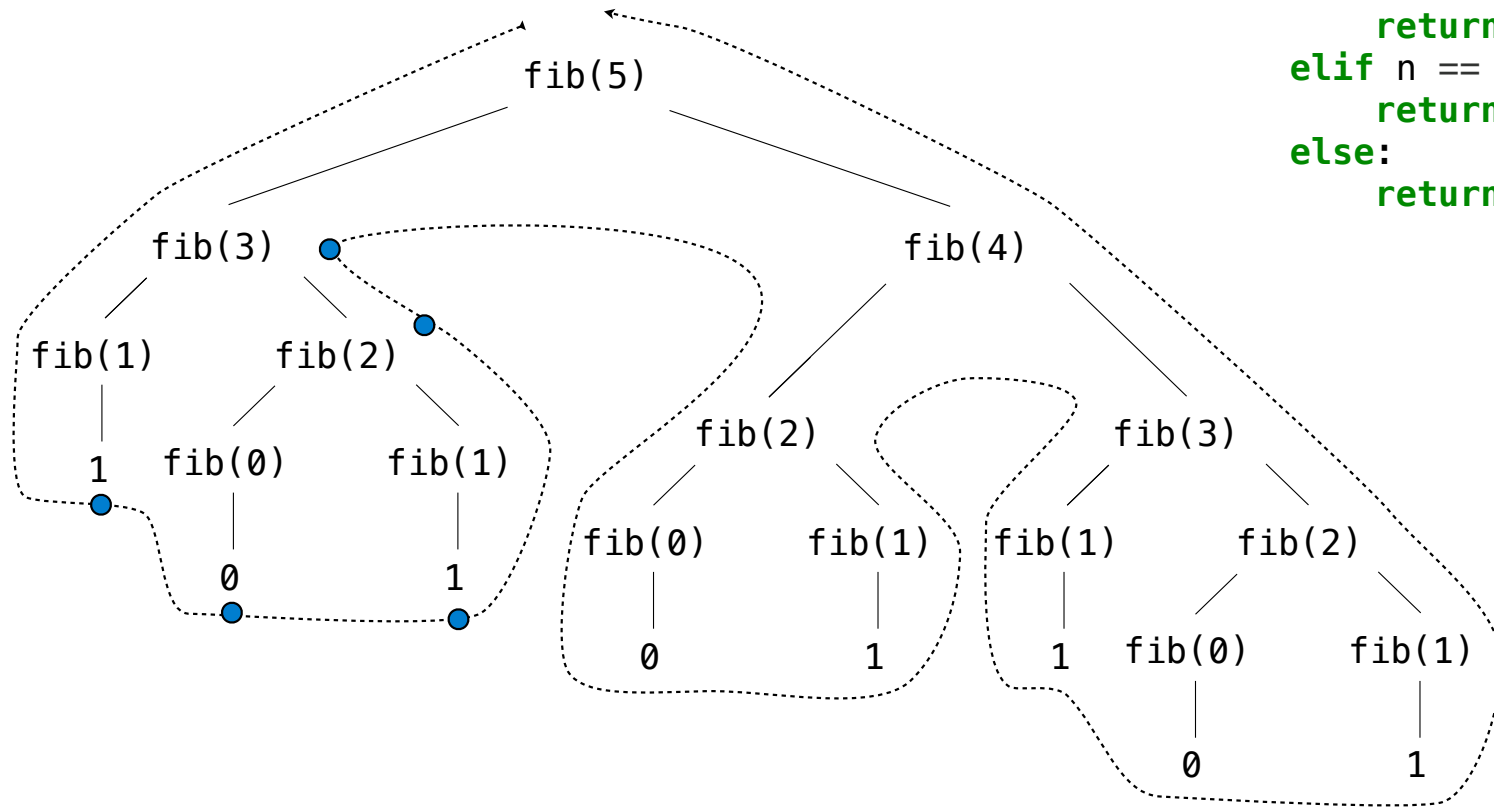


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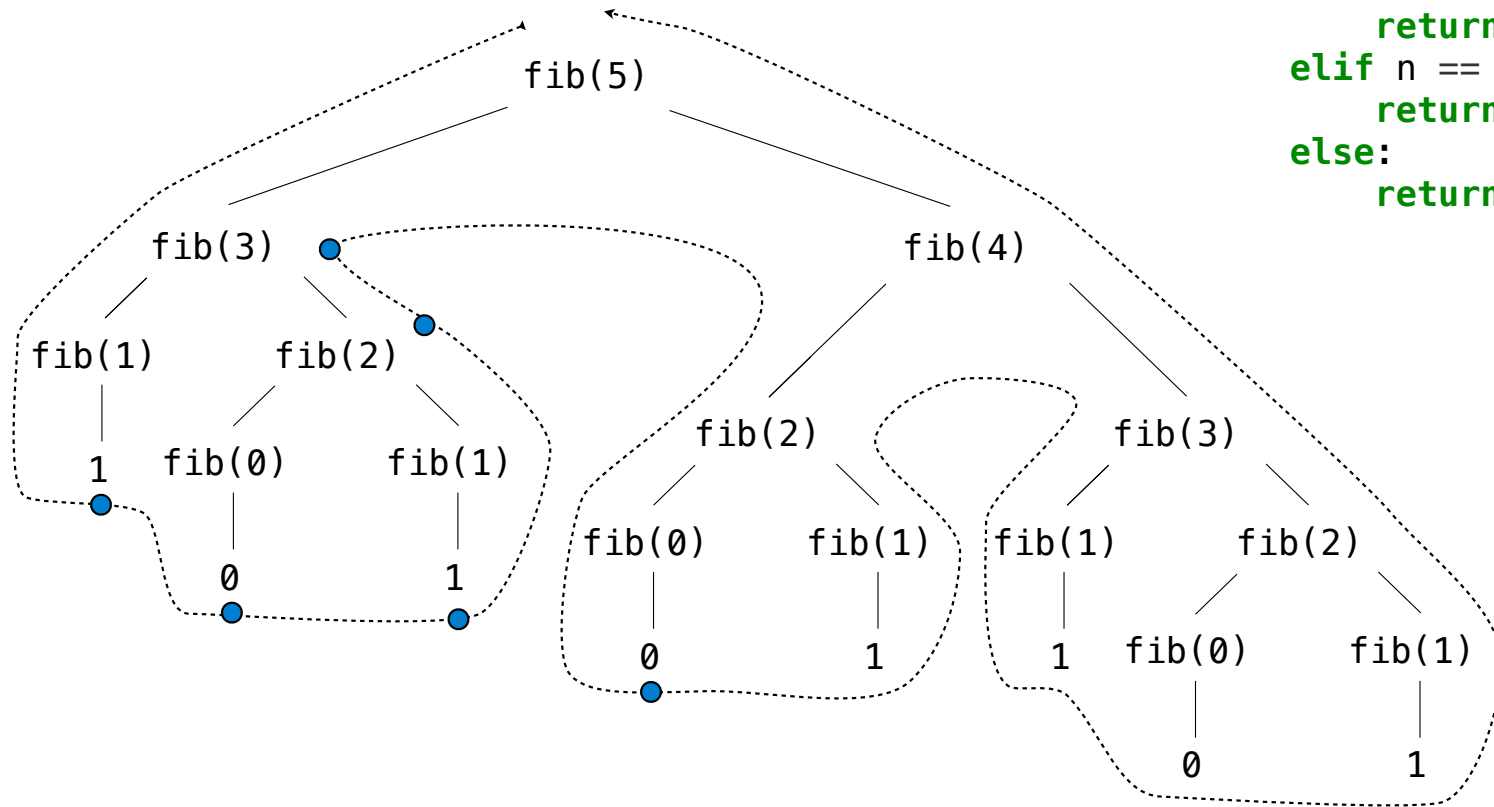


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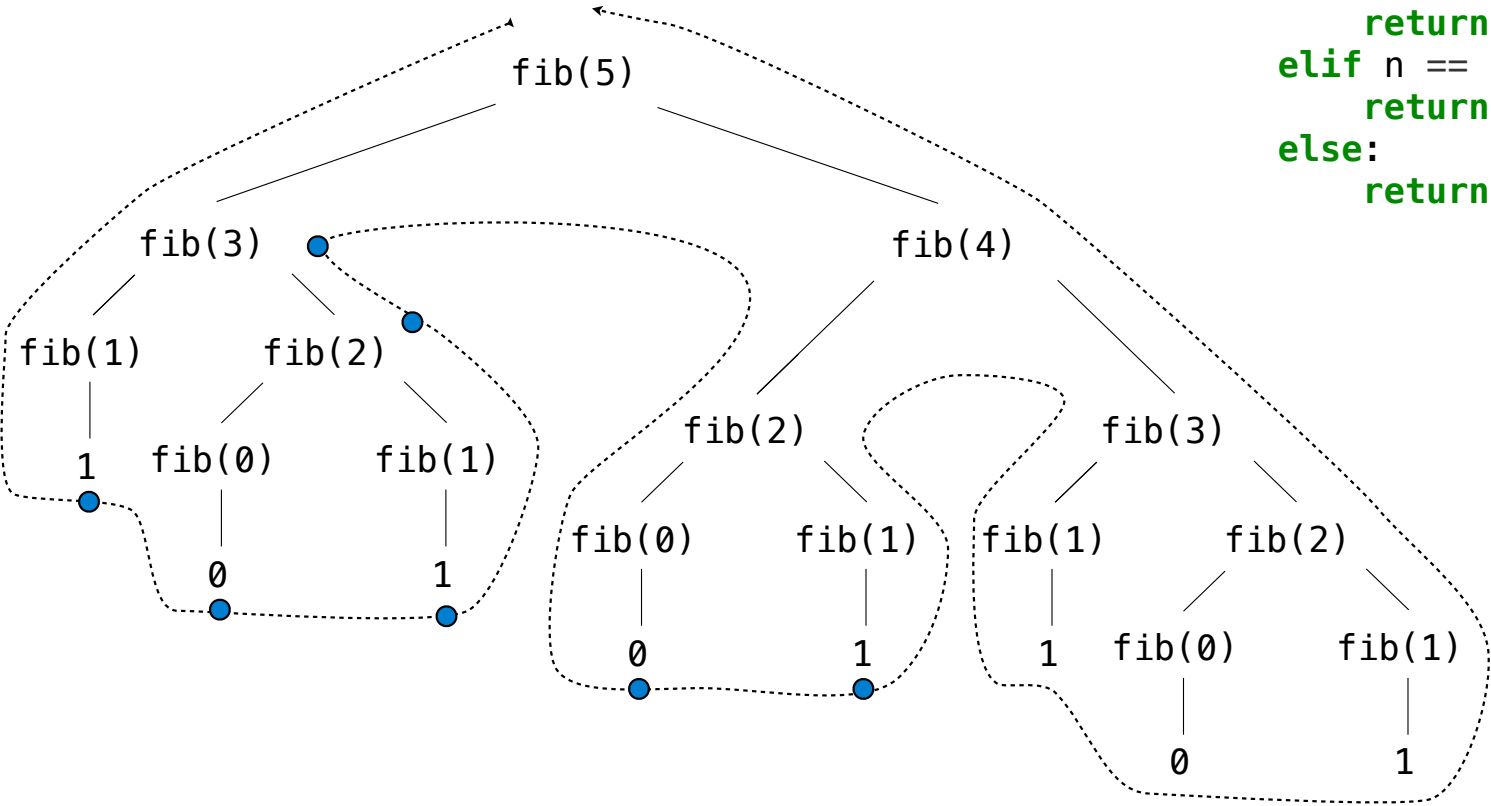
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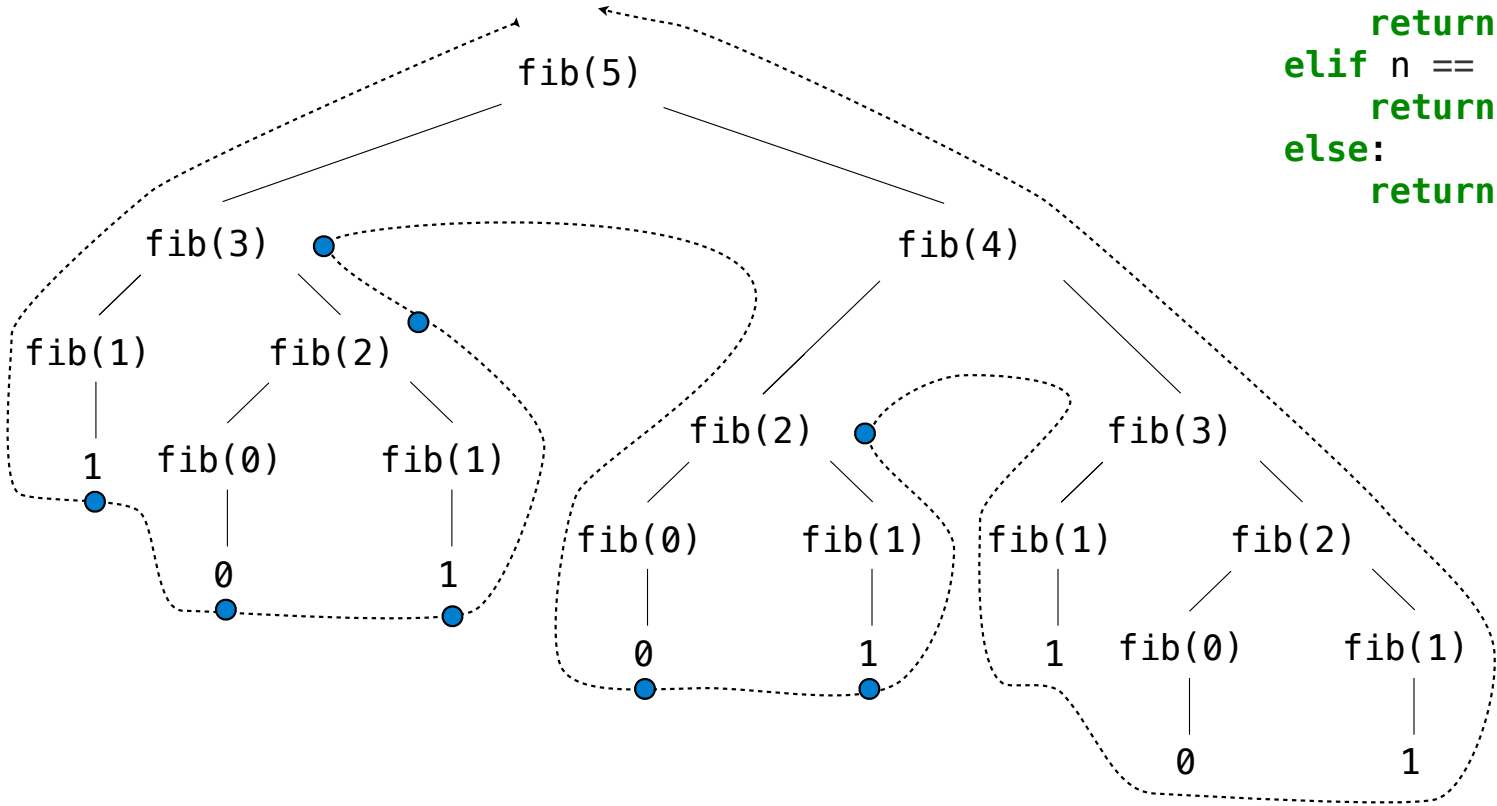
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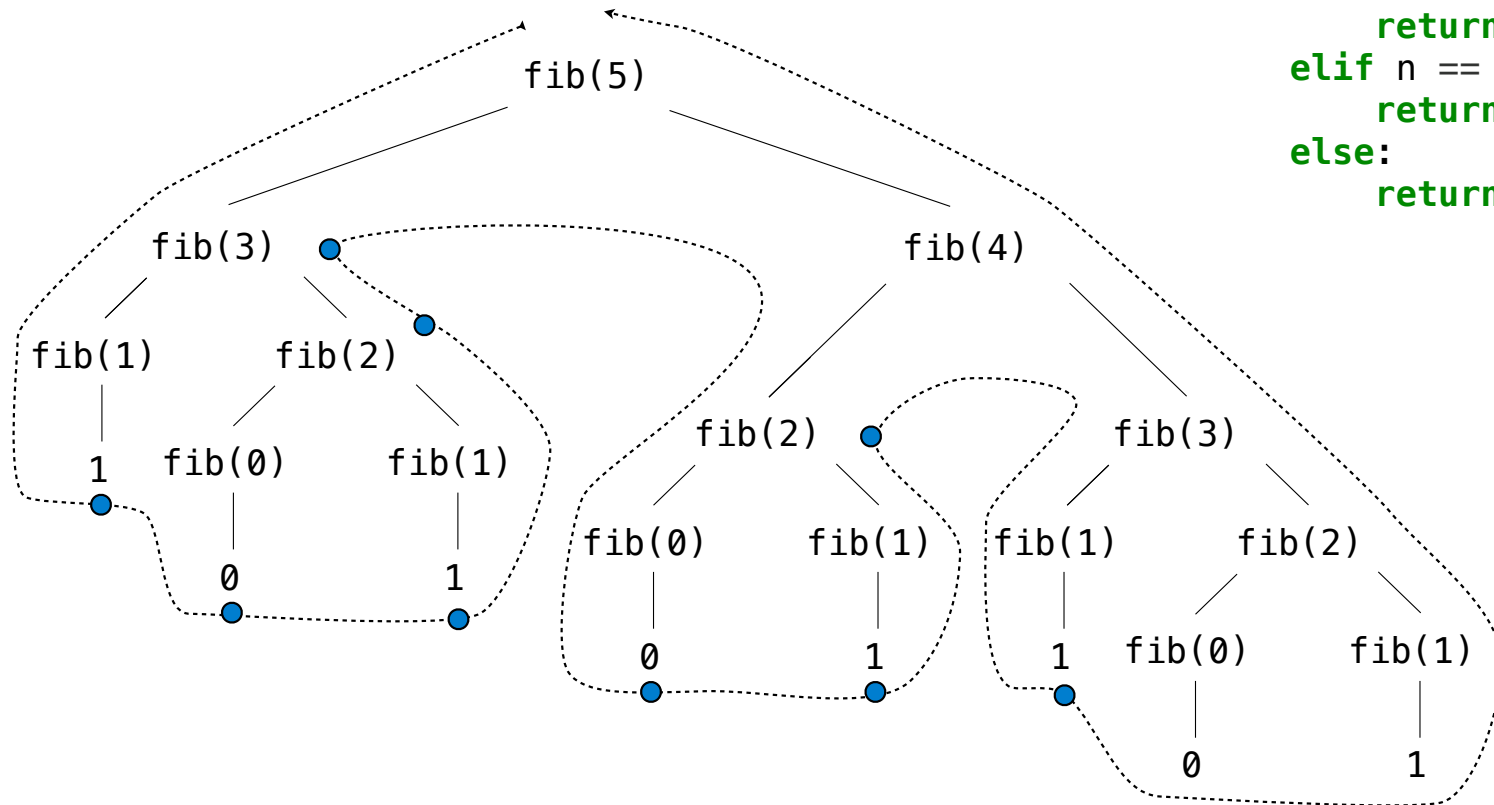


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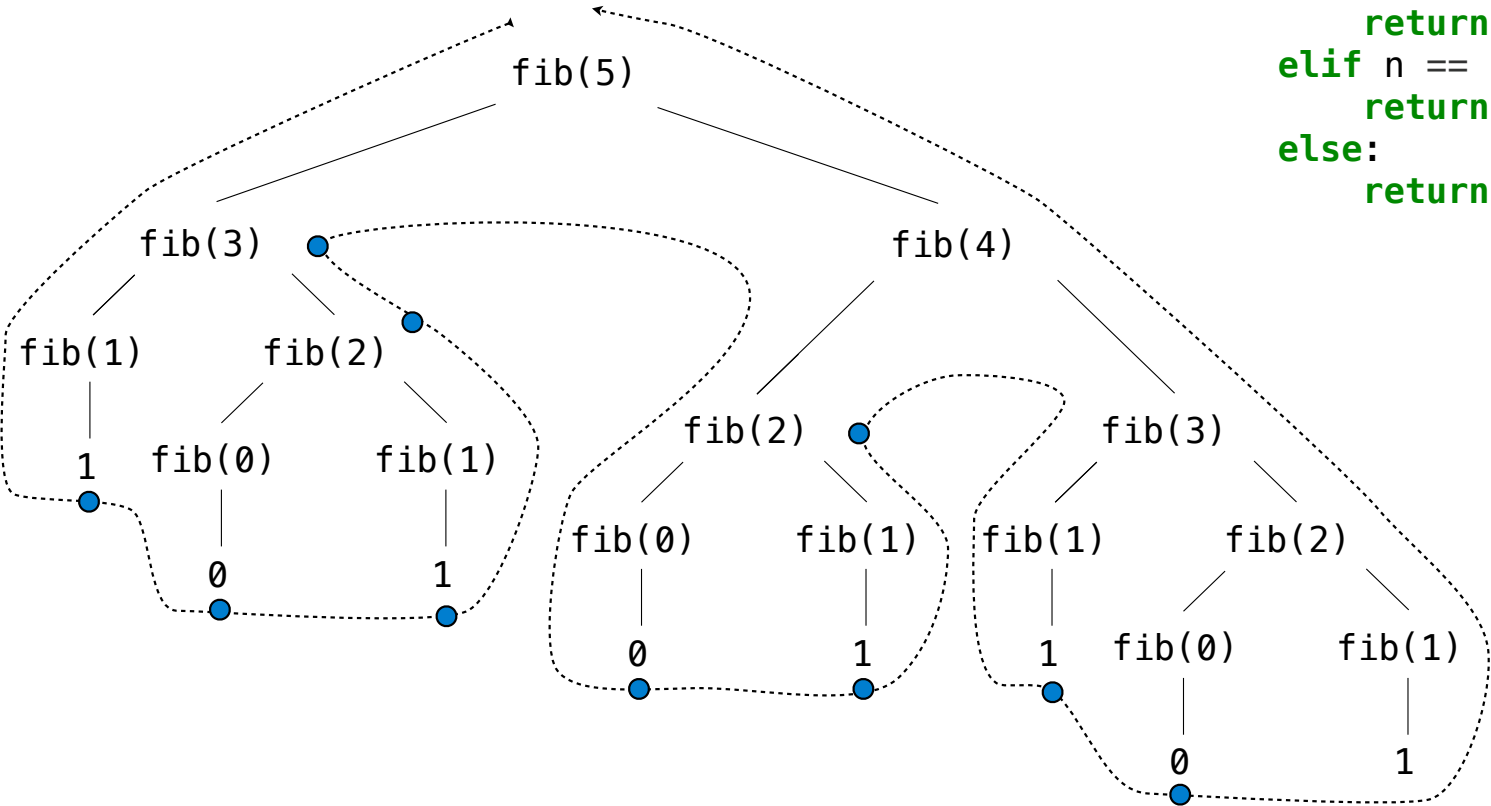
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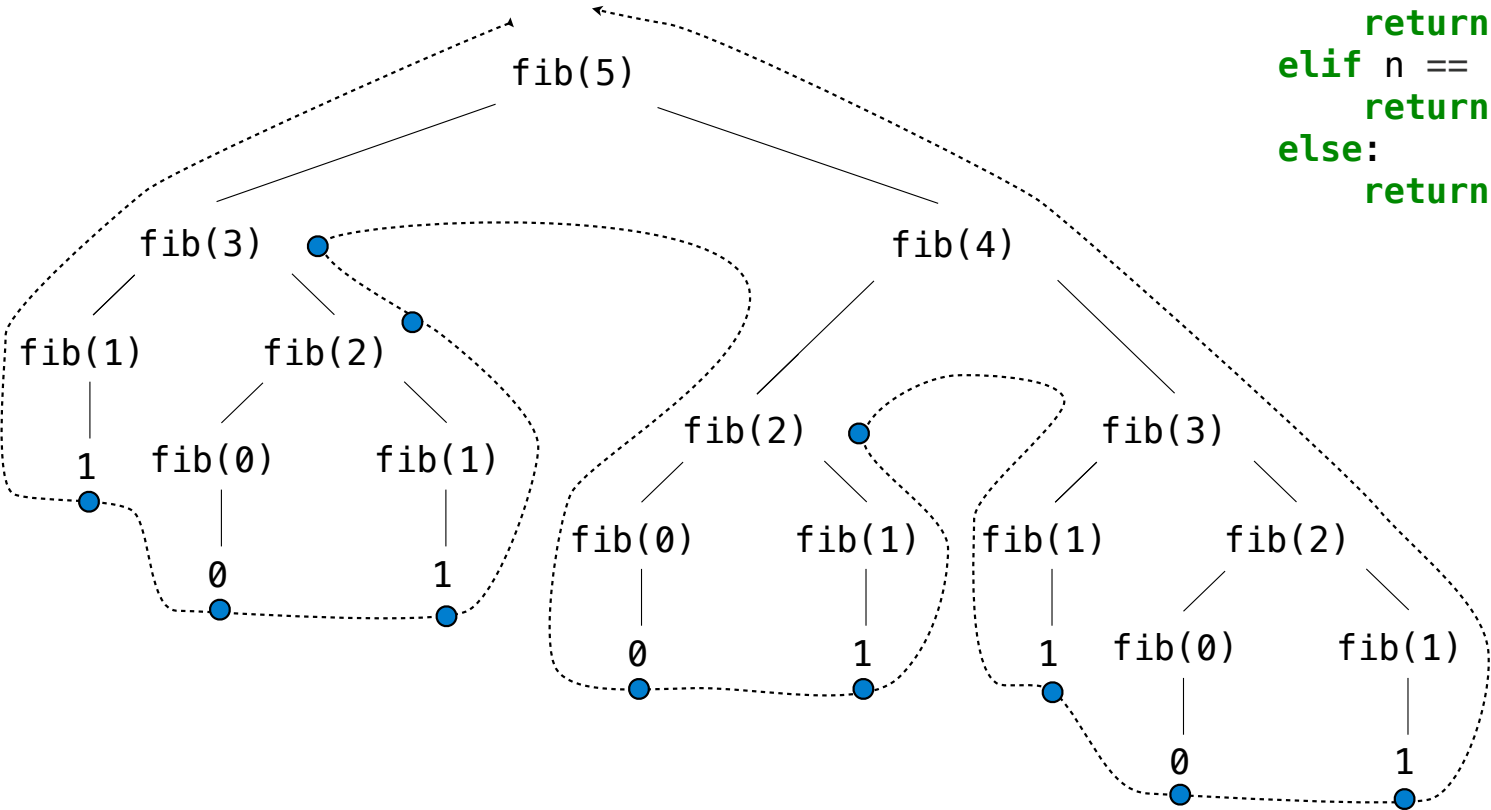
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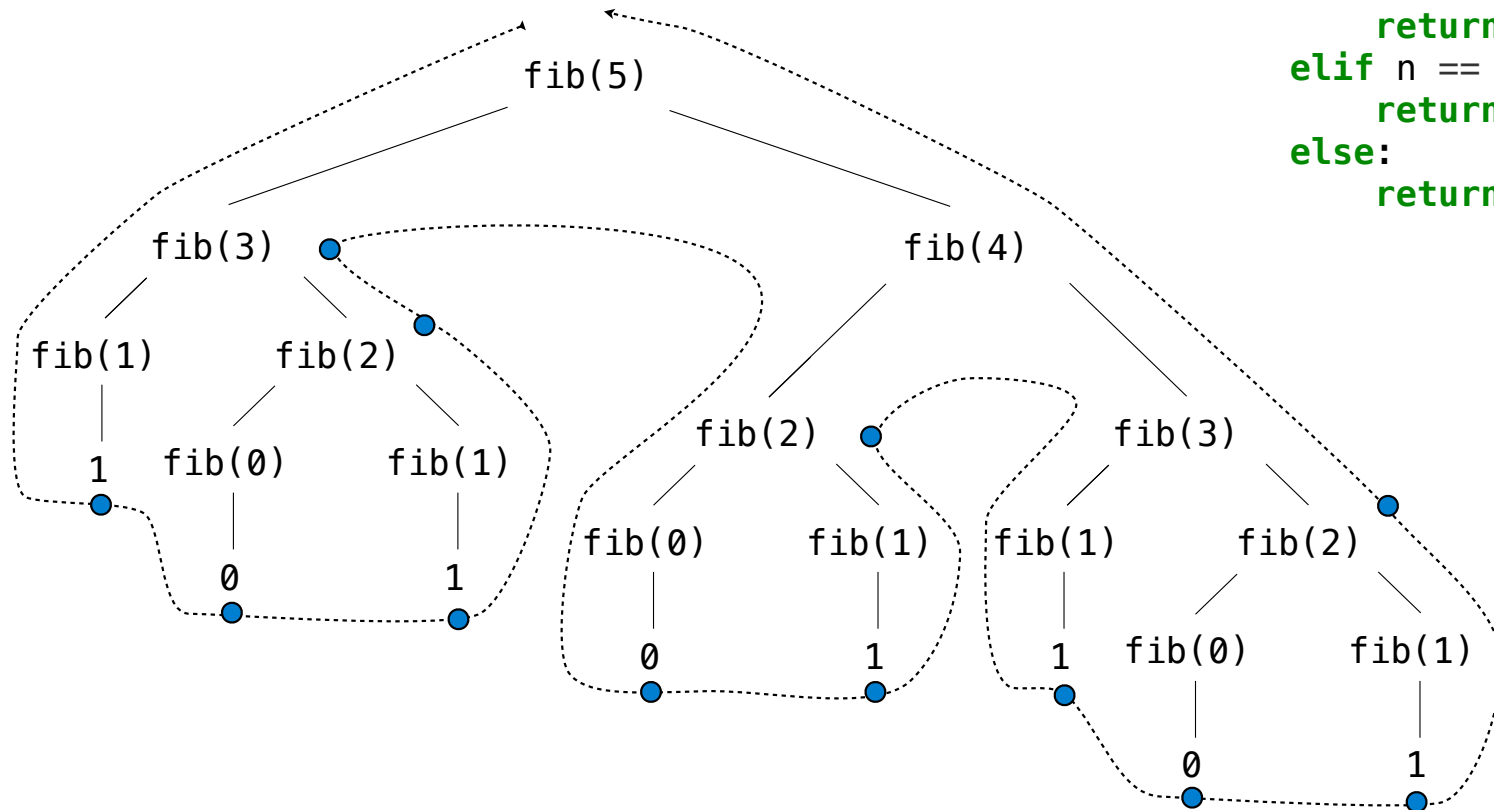
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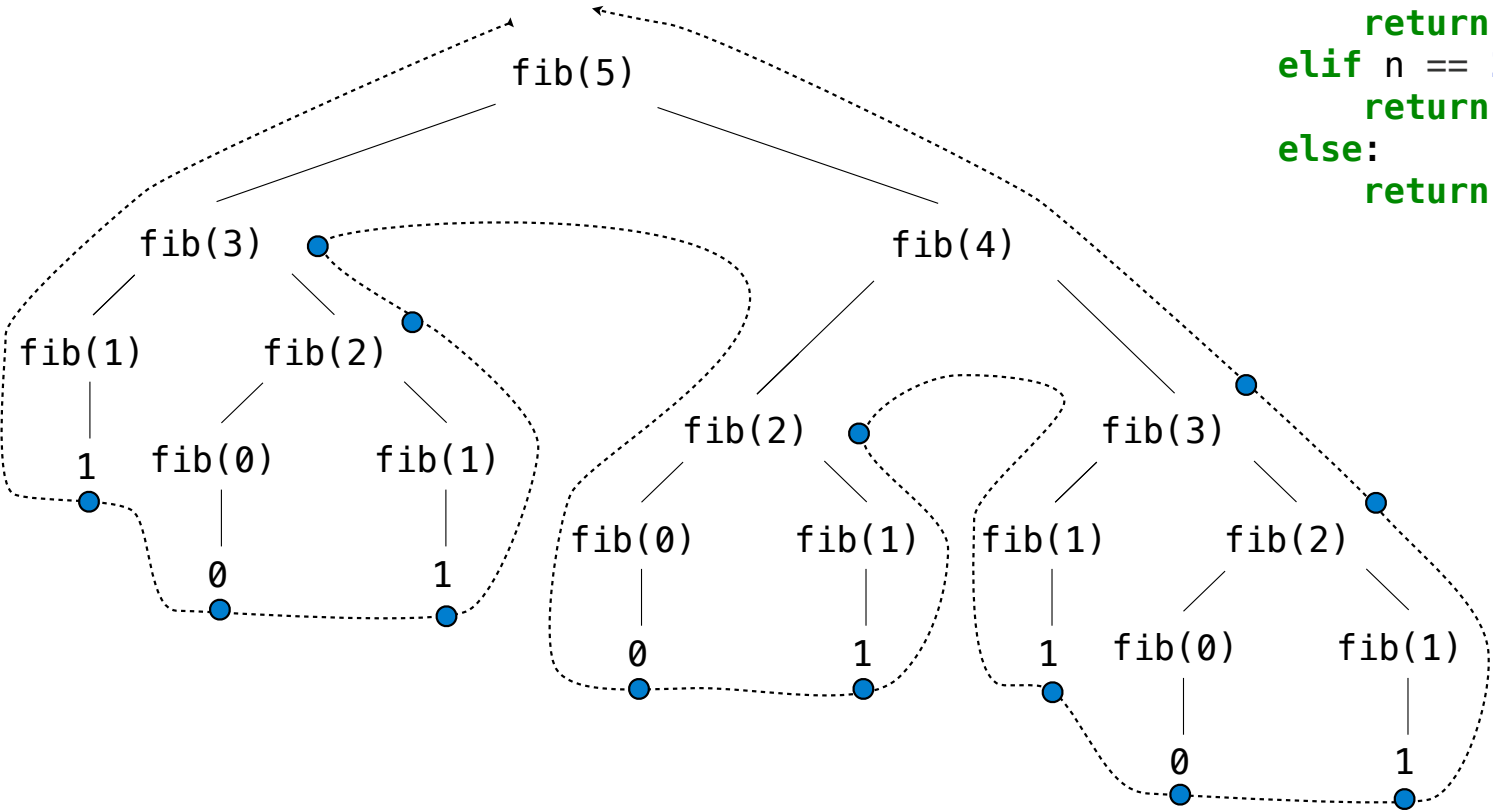
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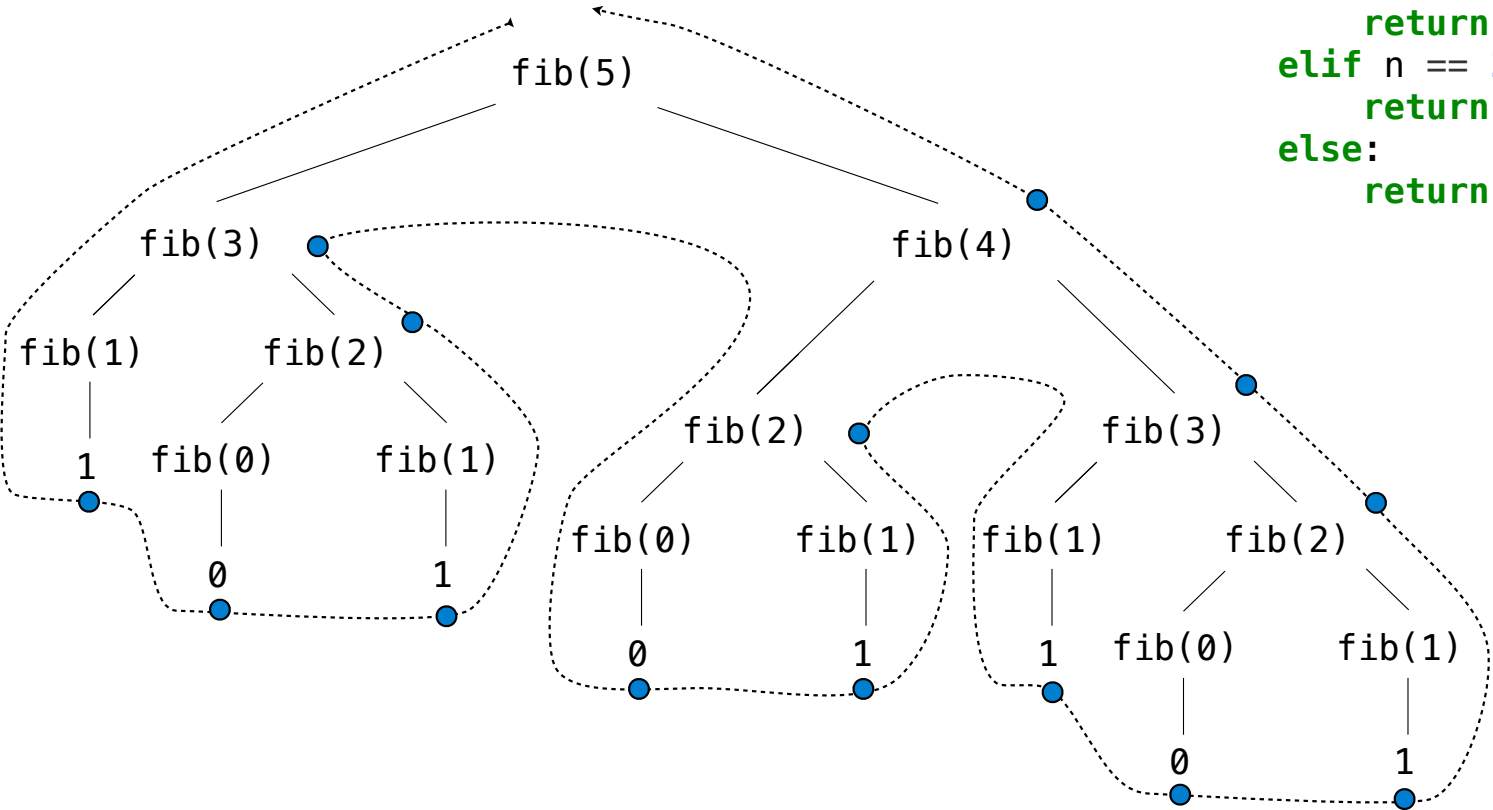


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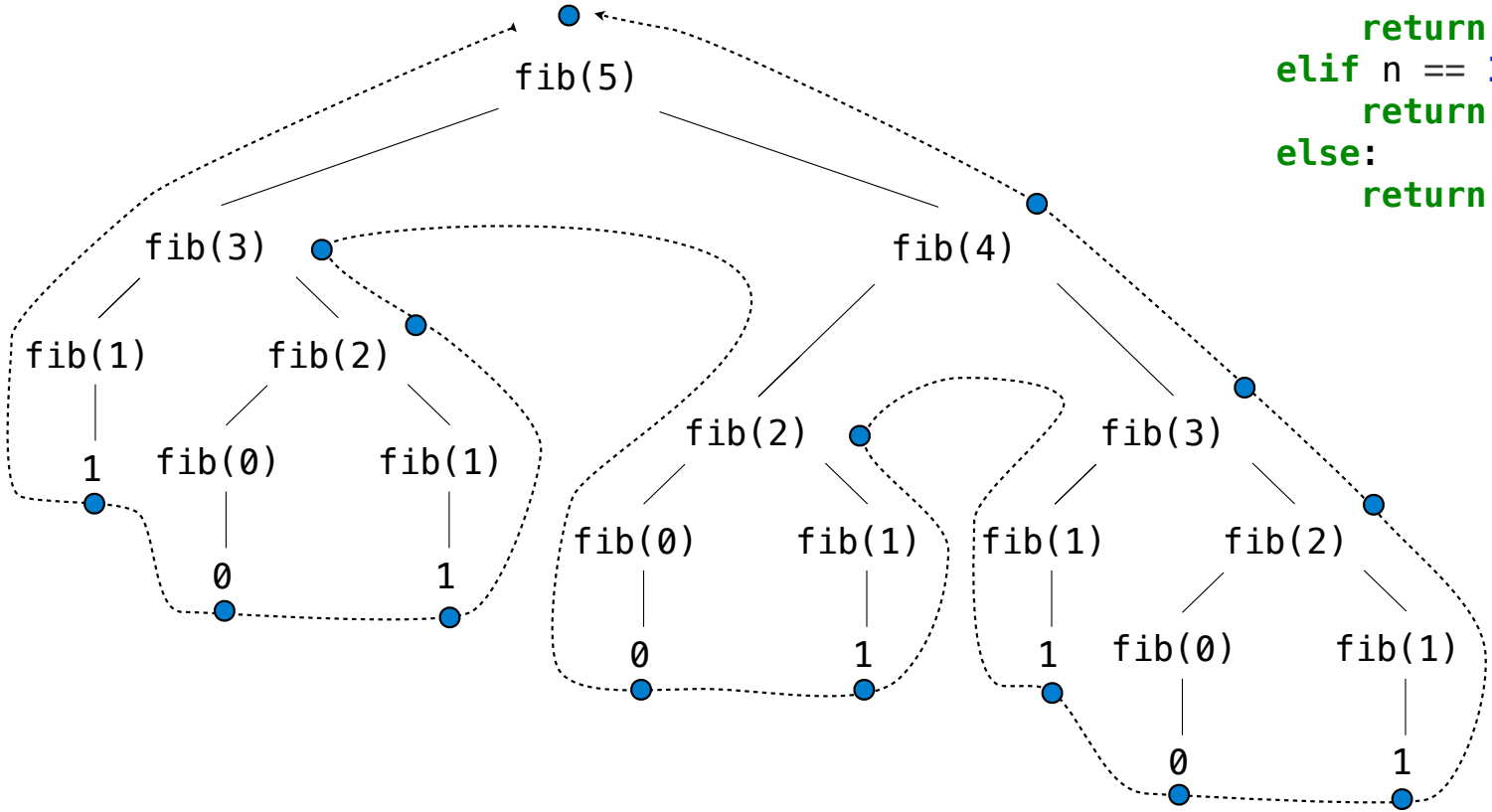
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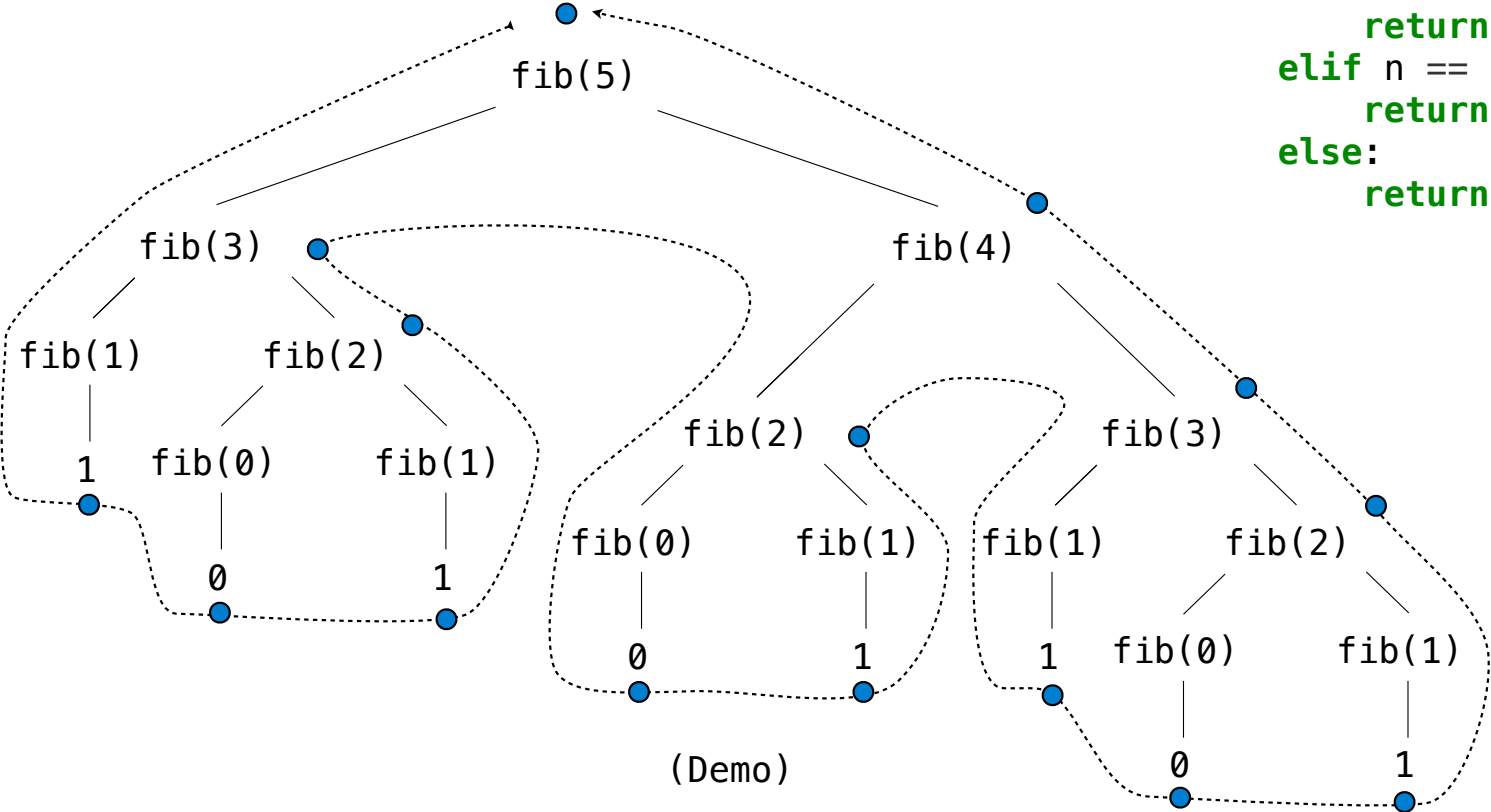
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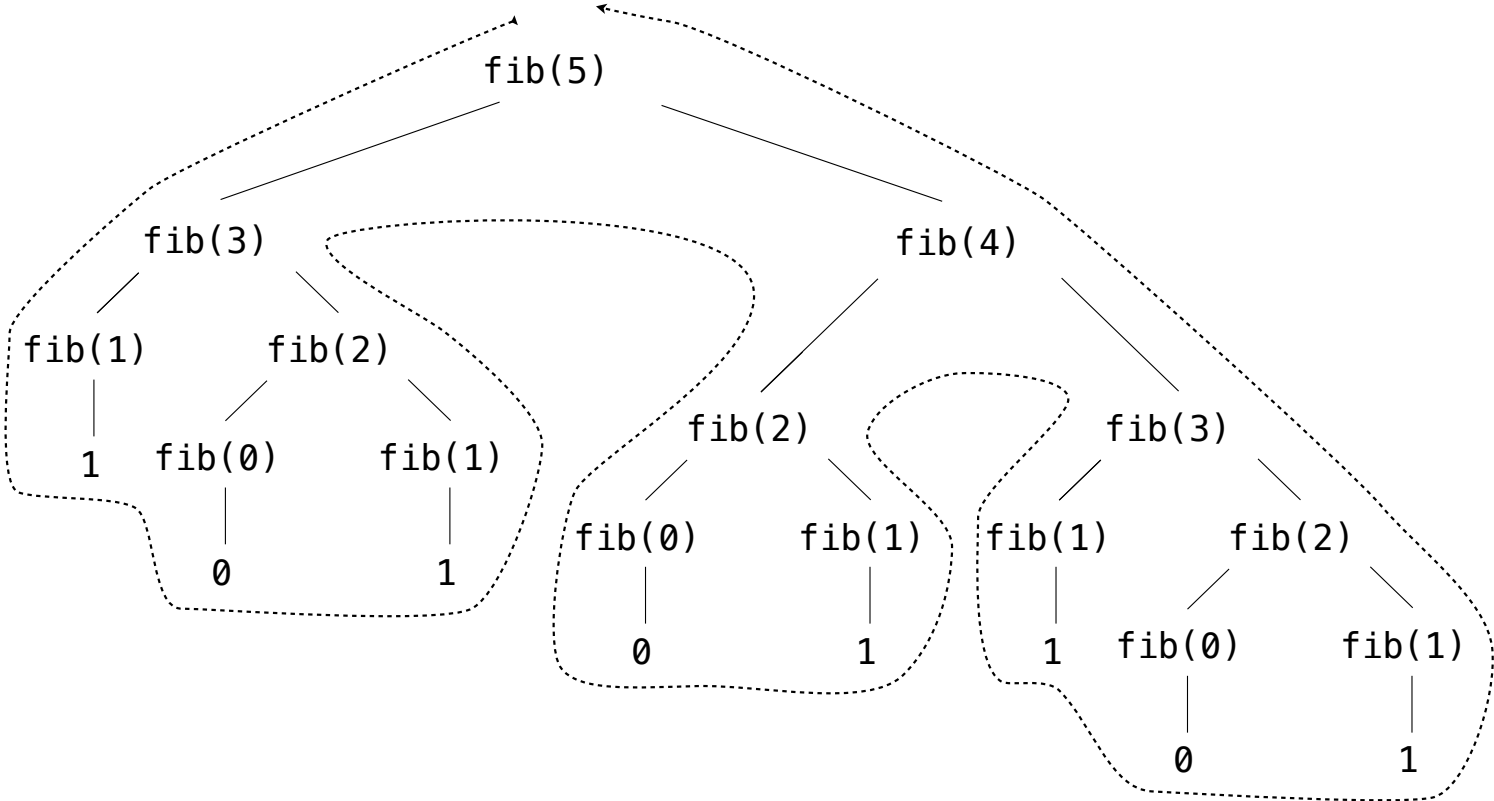
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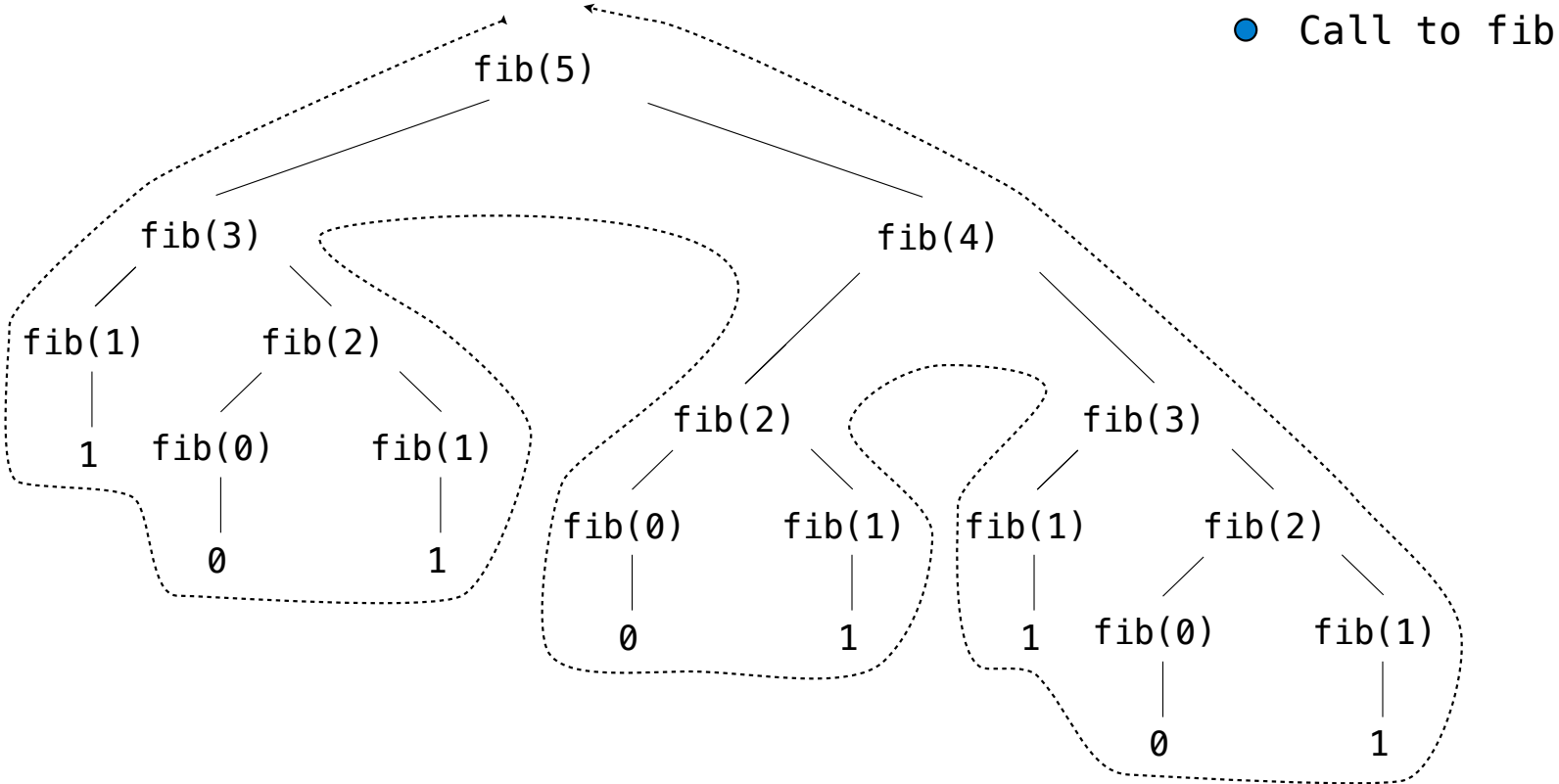
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(Demo)

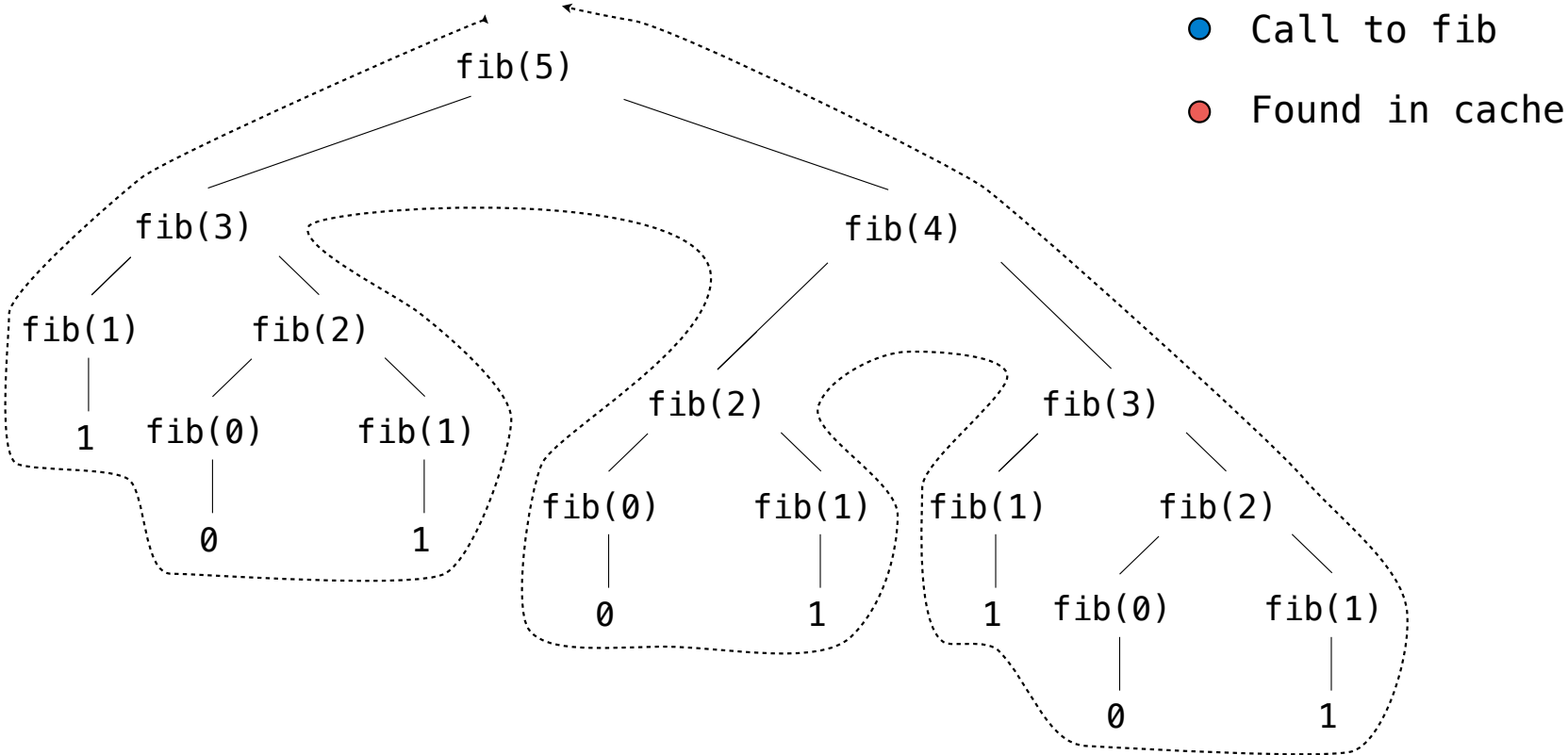
Memoized Tree Recursion



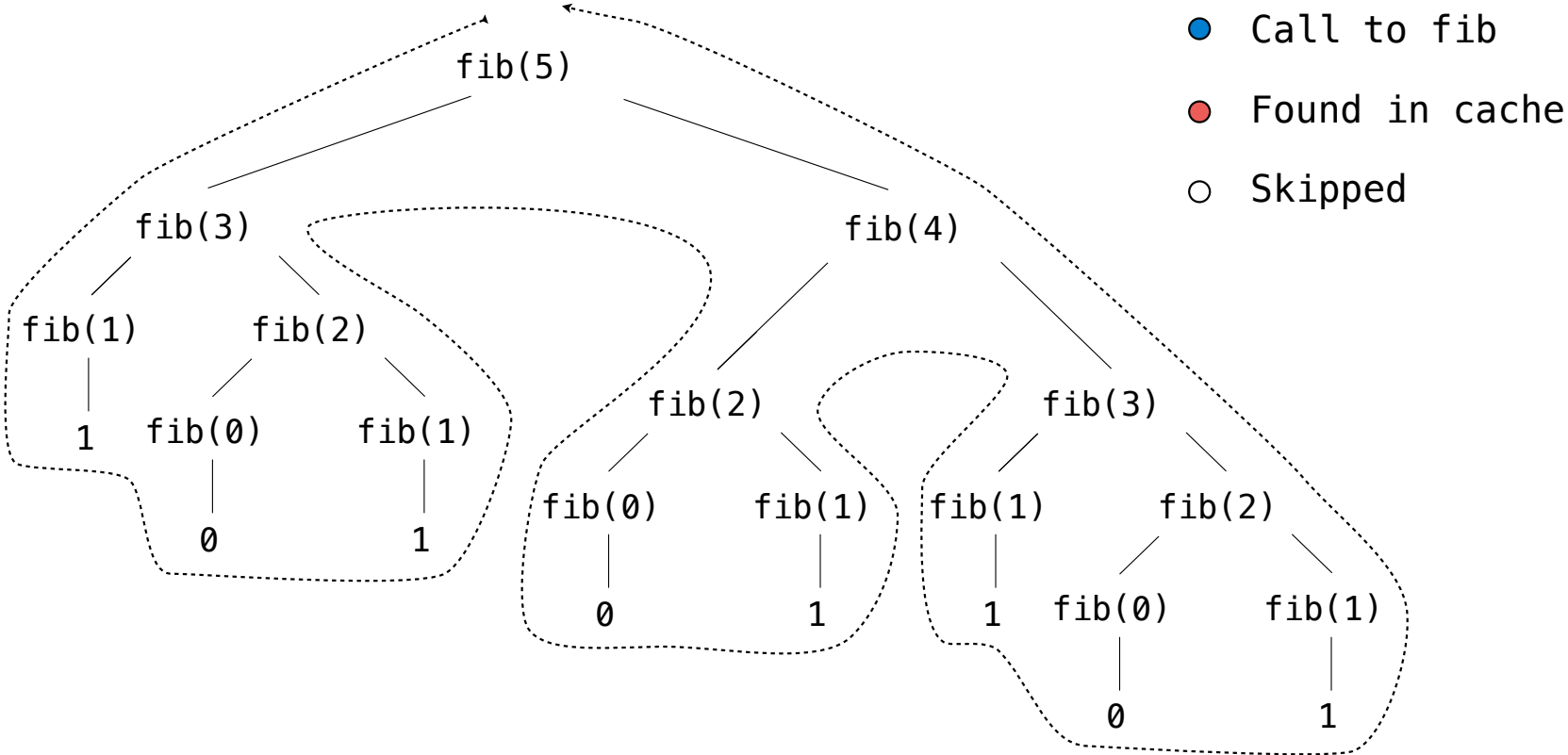
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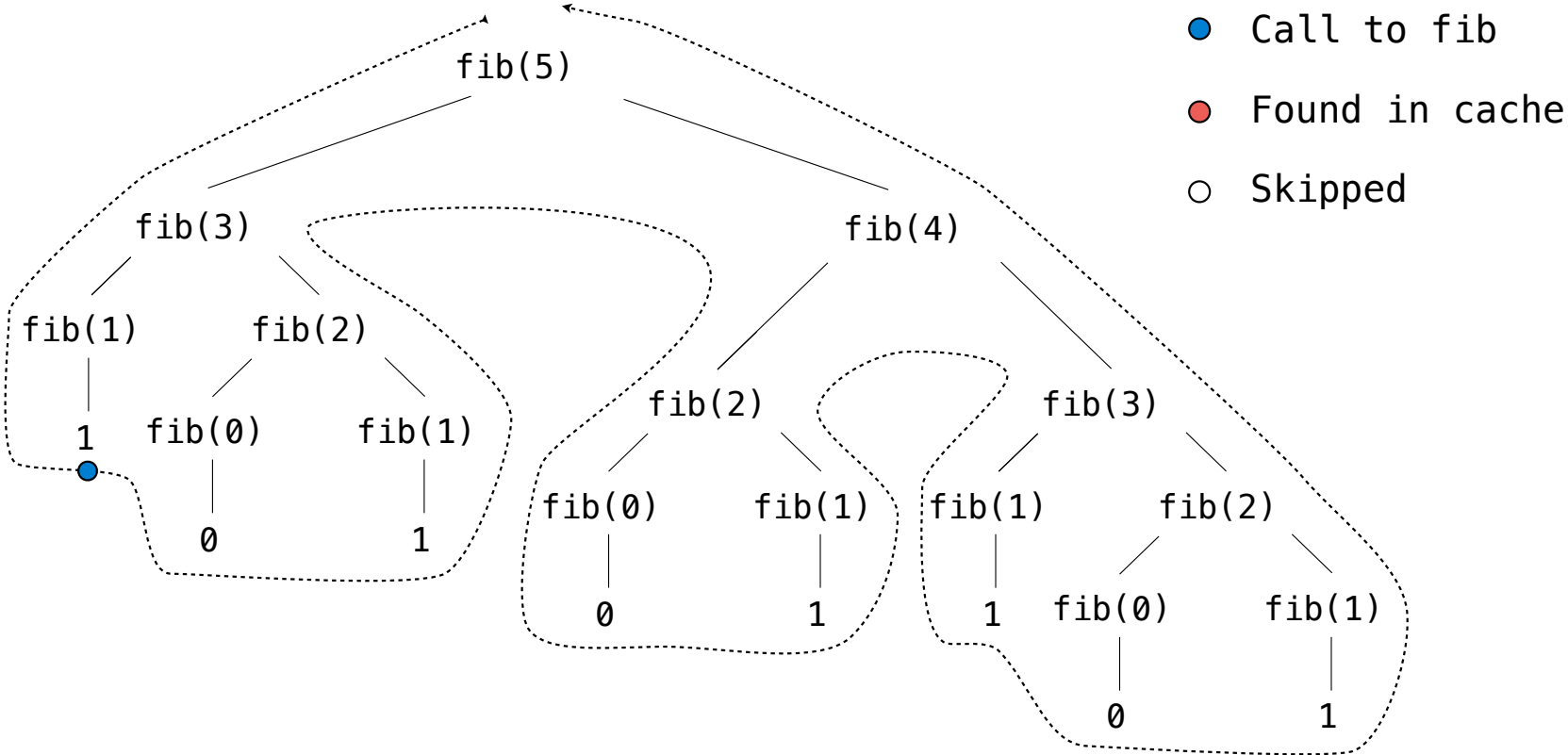
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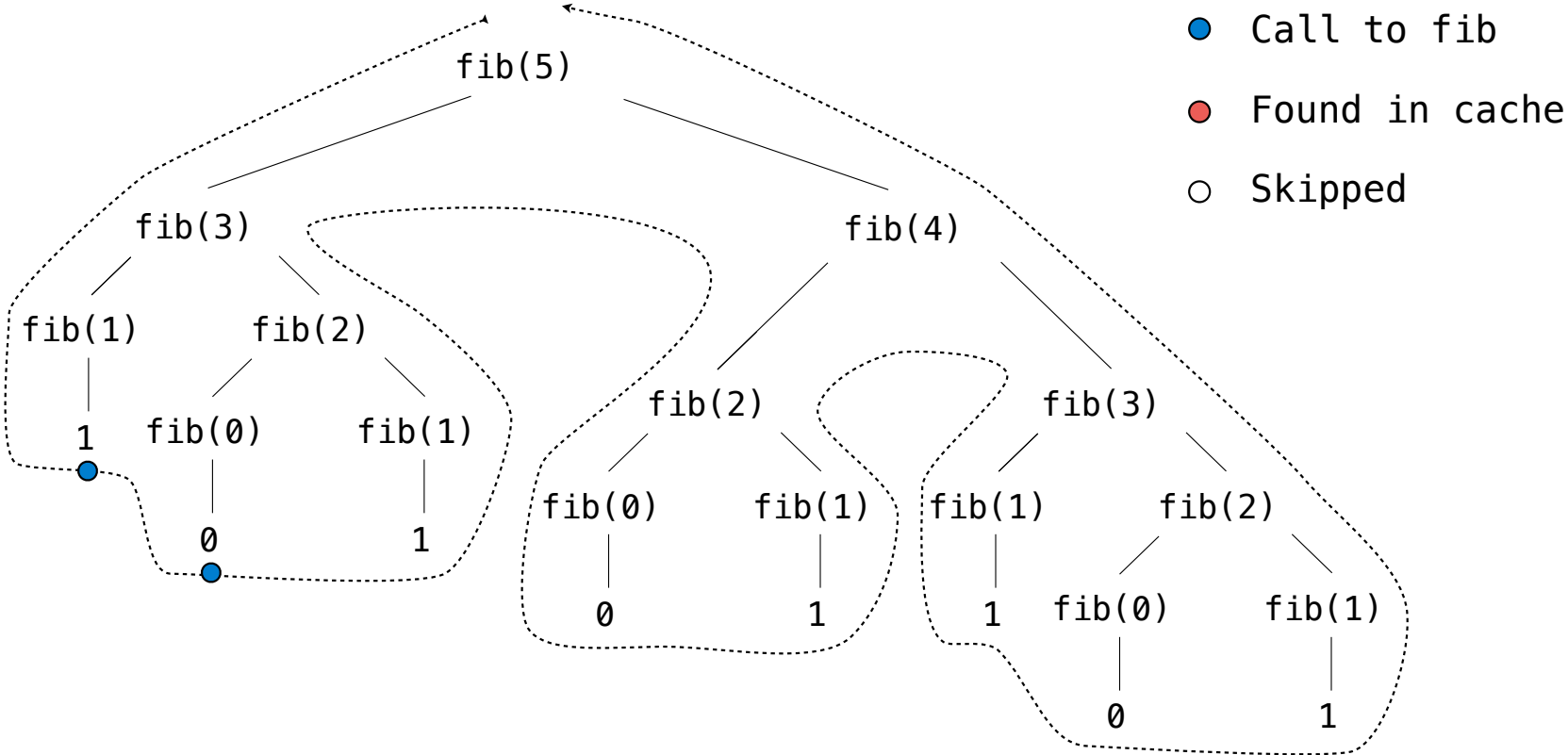
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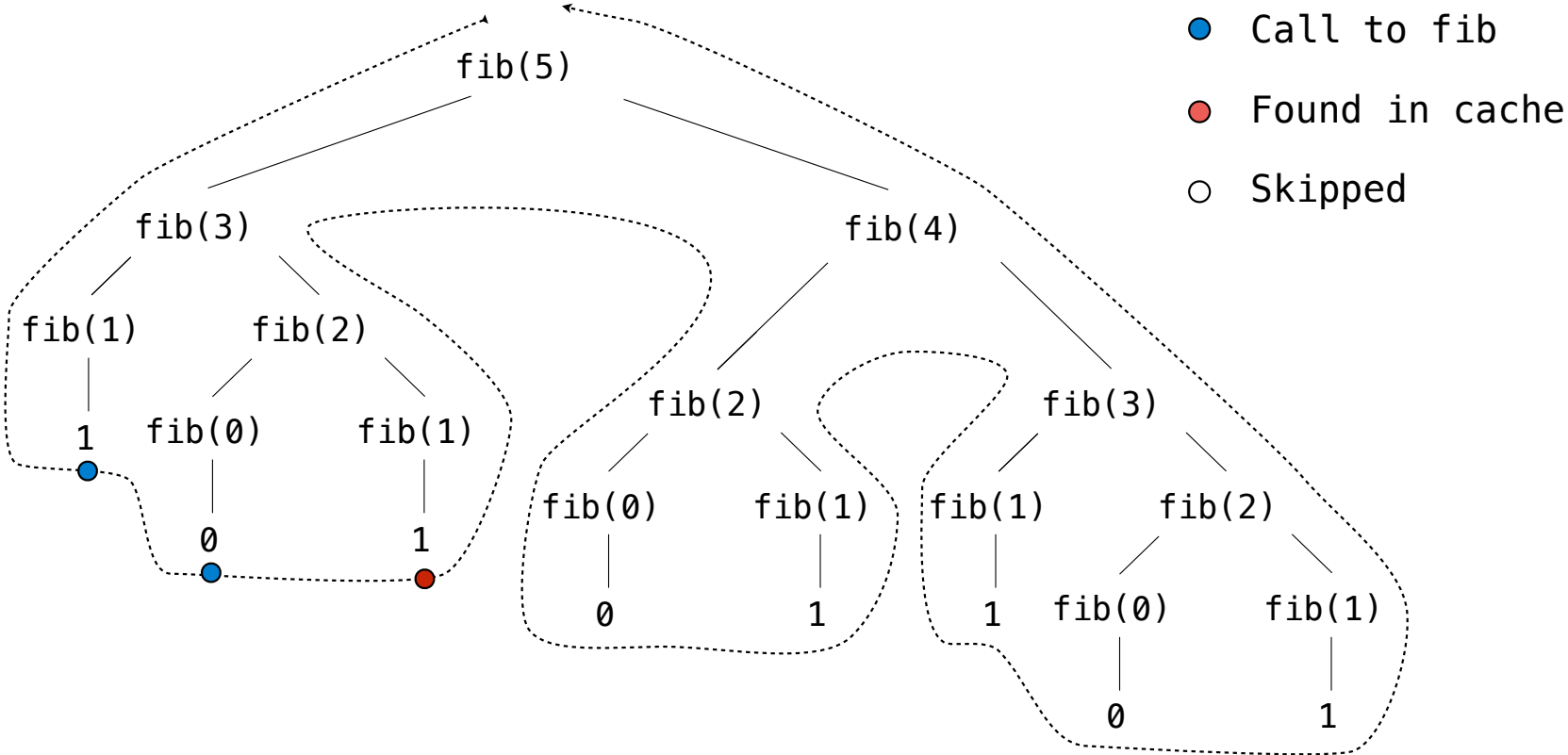
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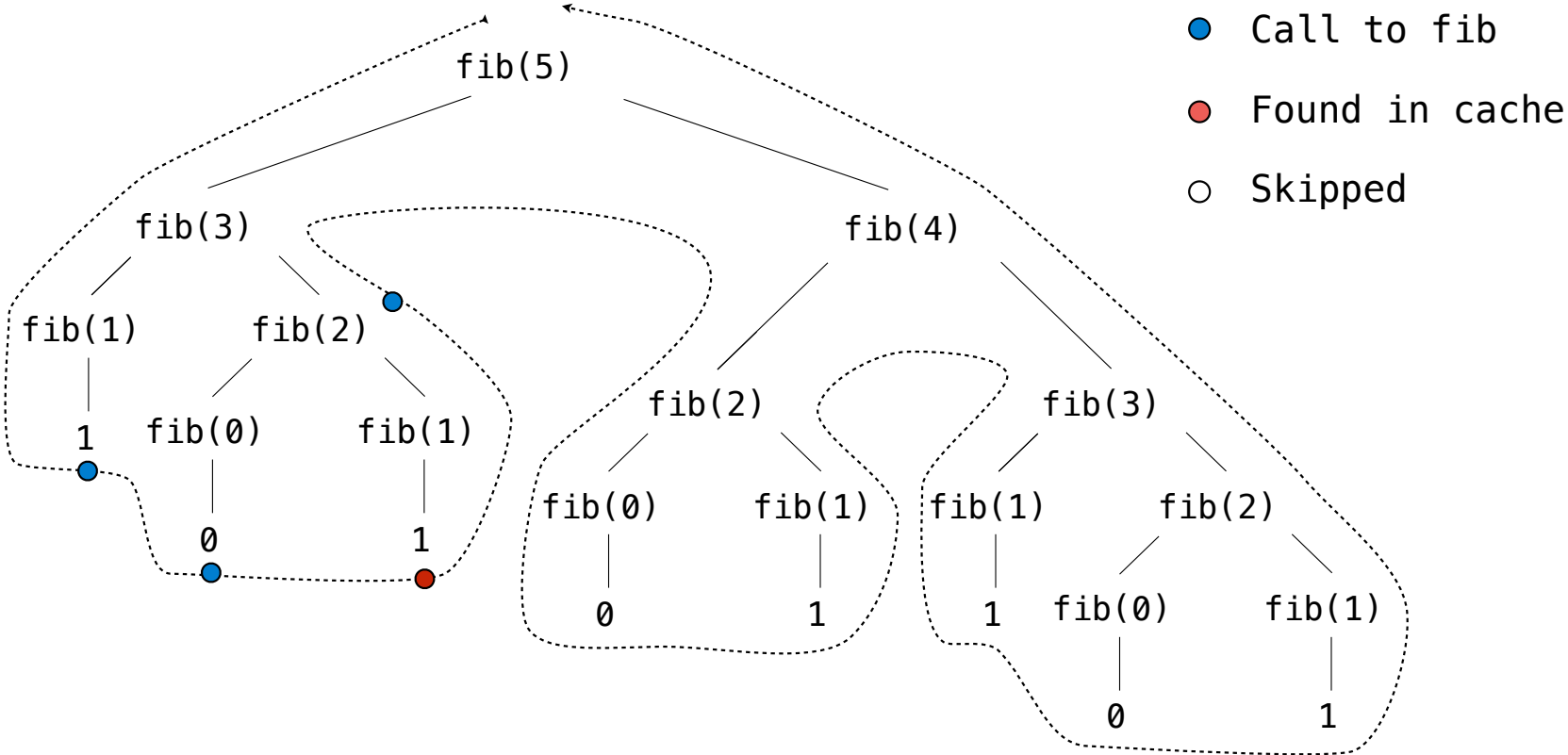
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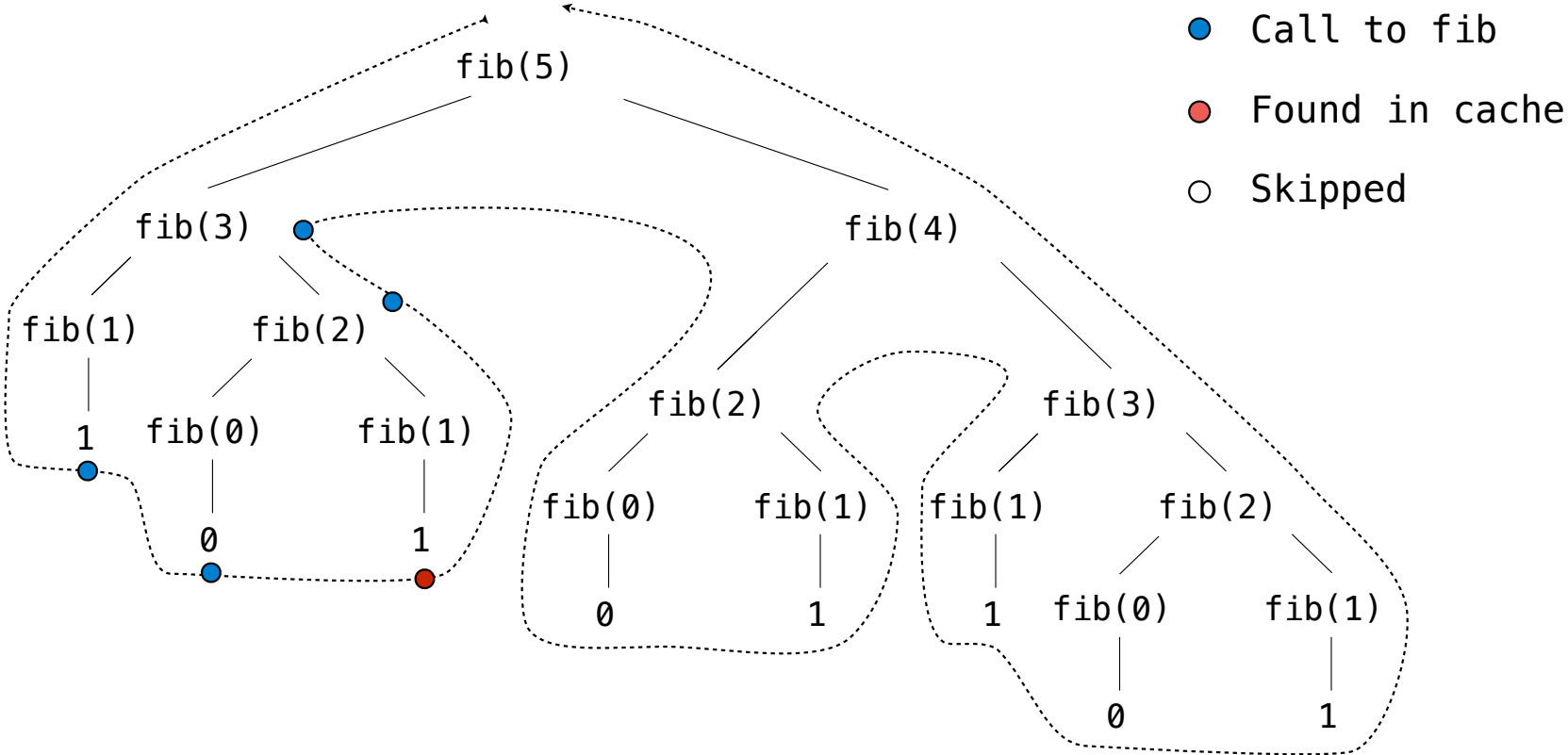
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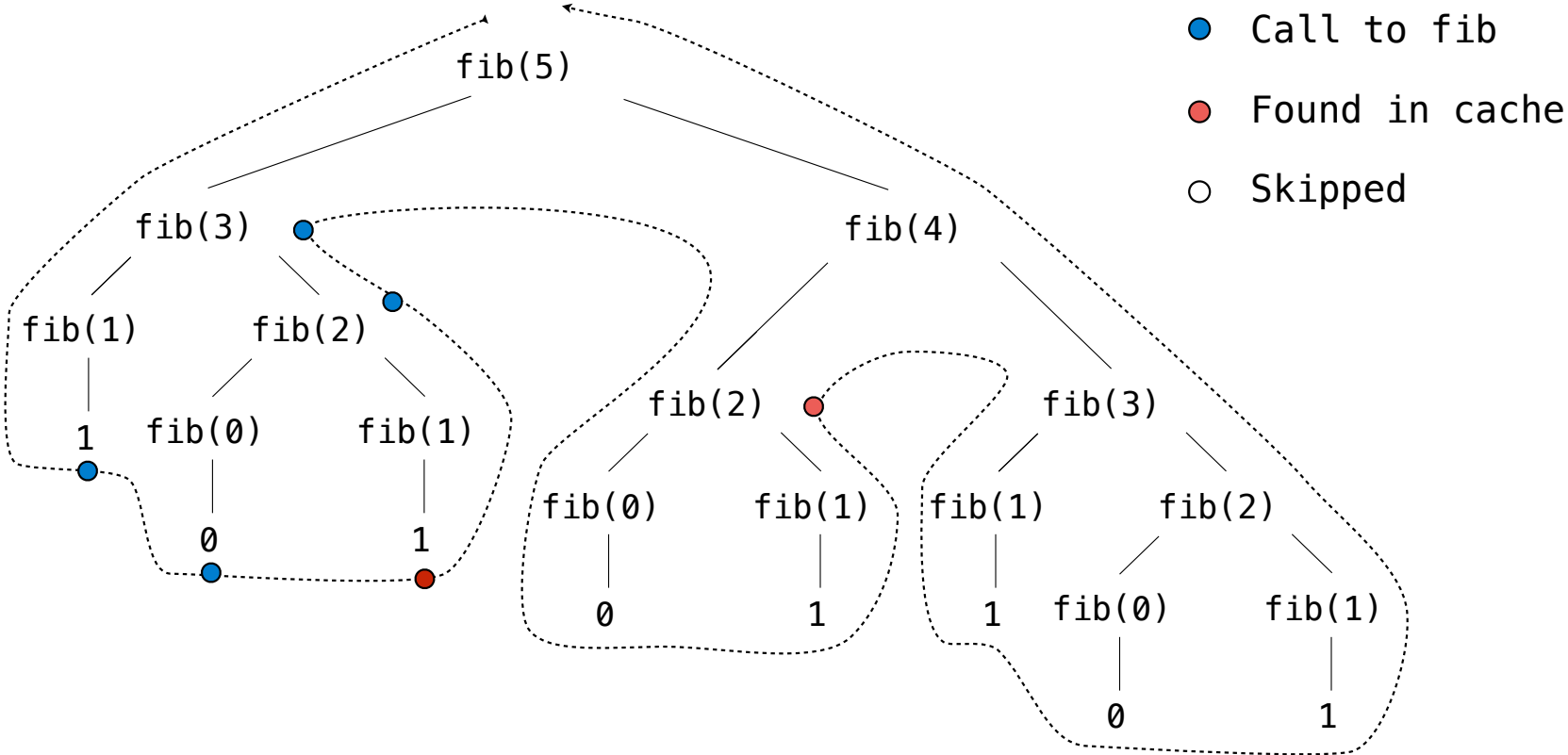
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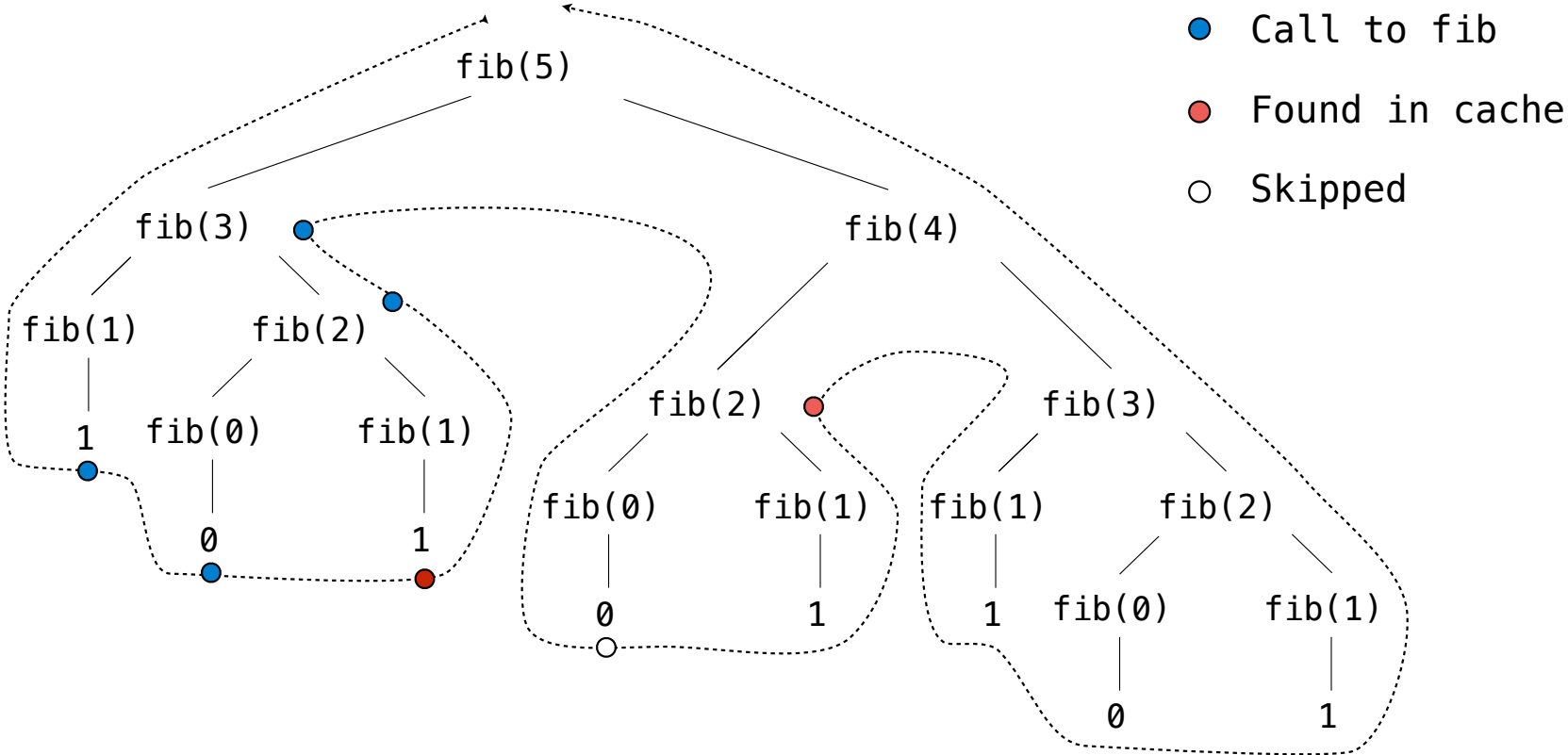
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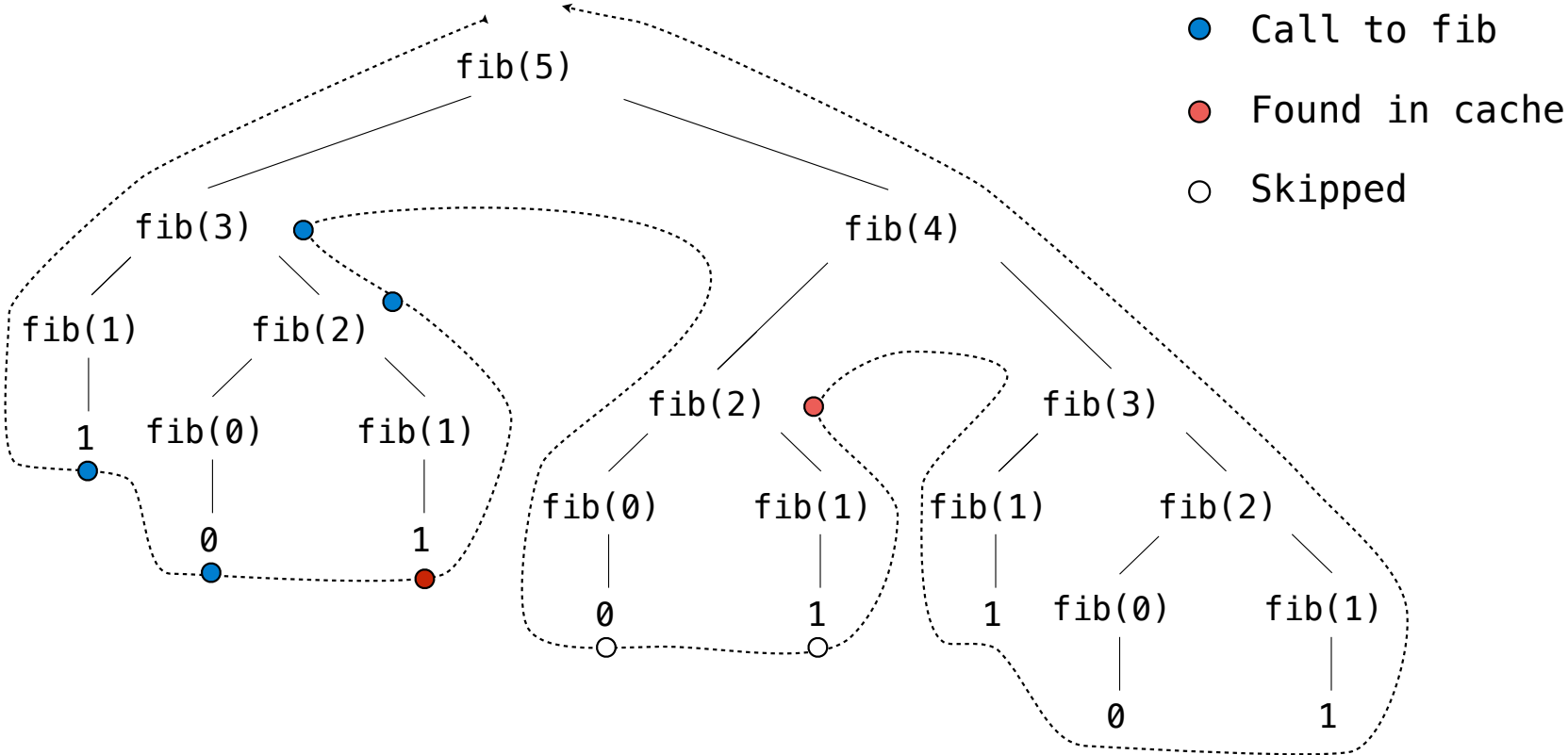
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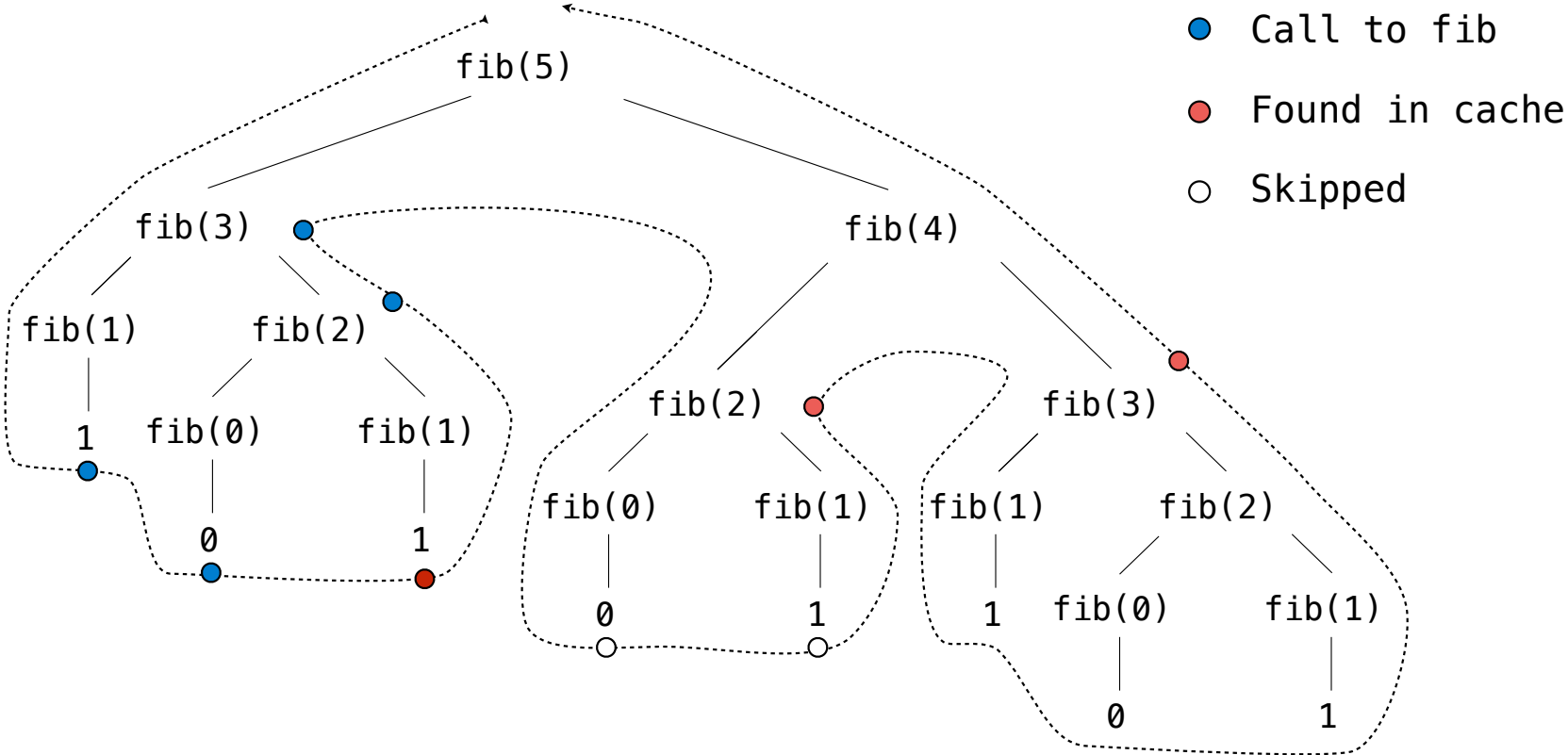
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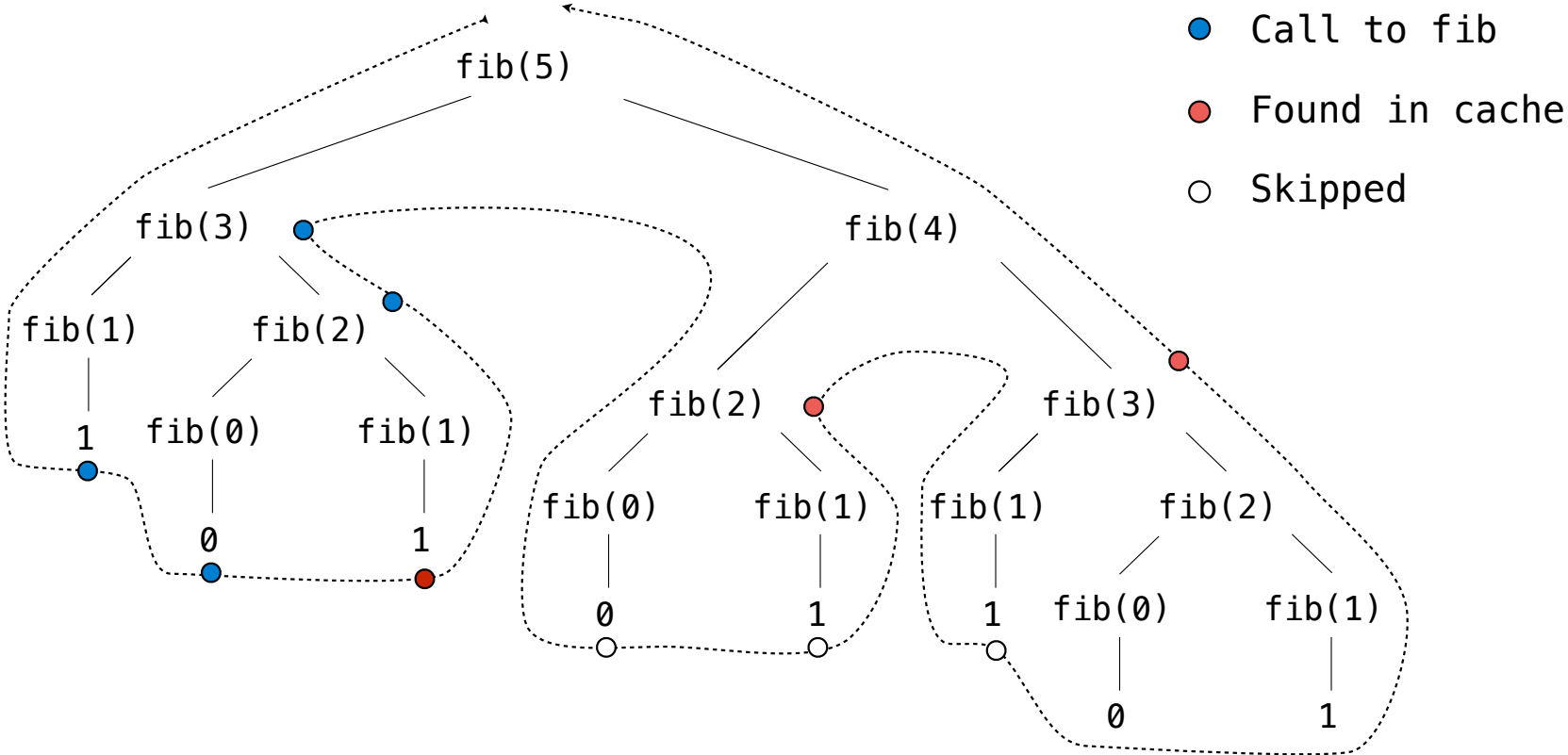
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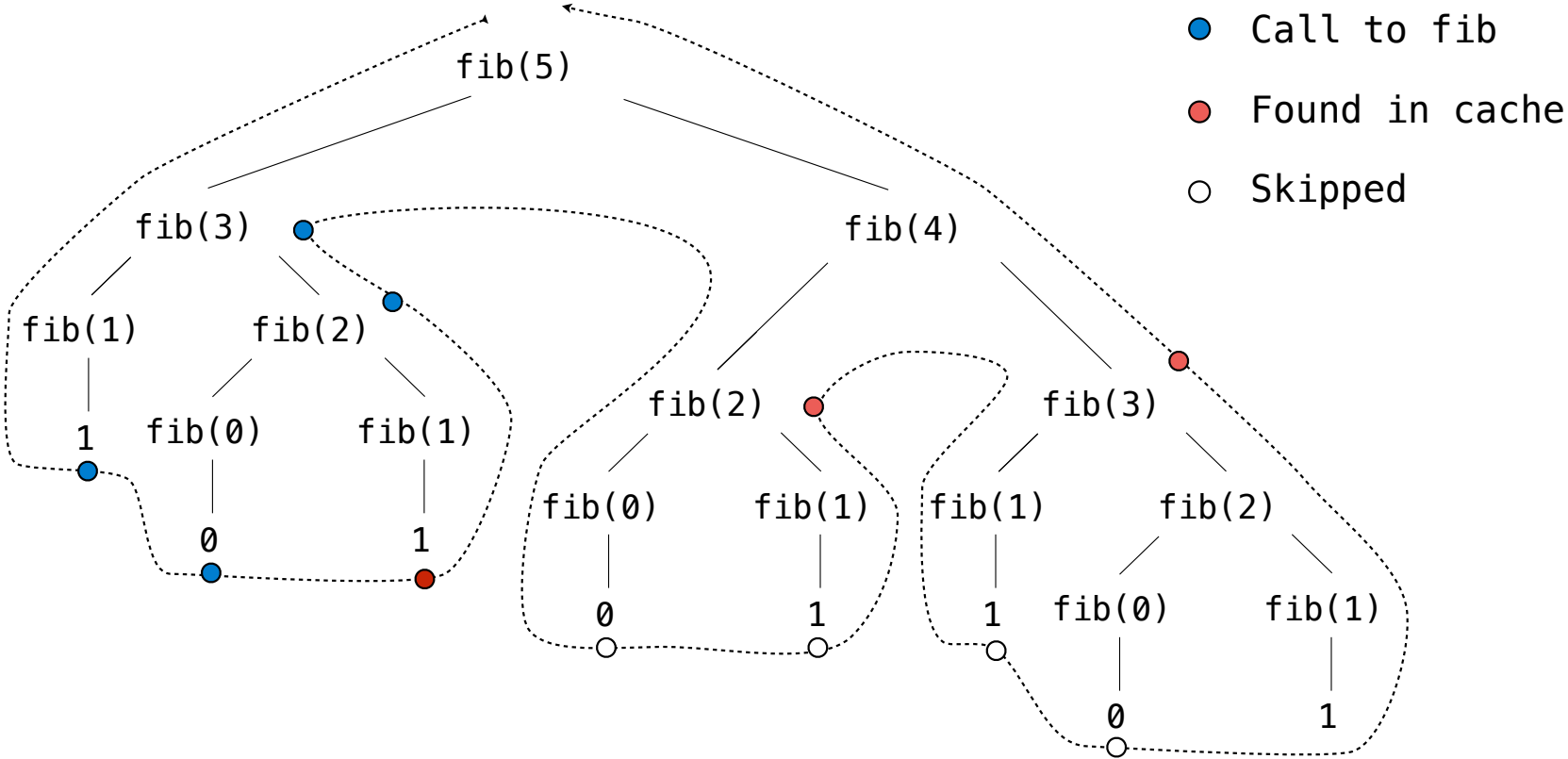
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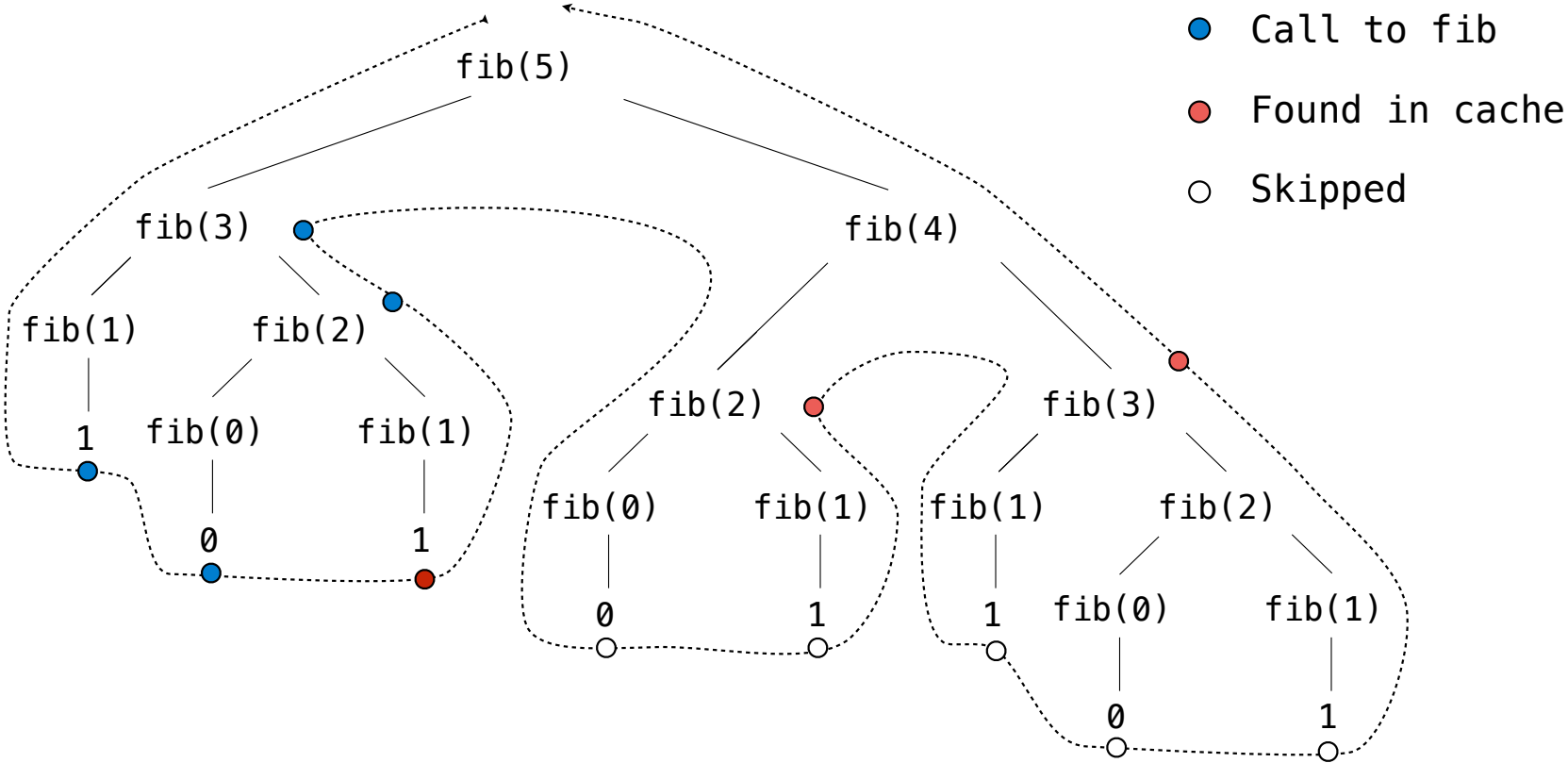
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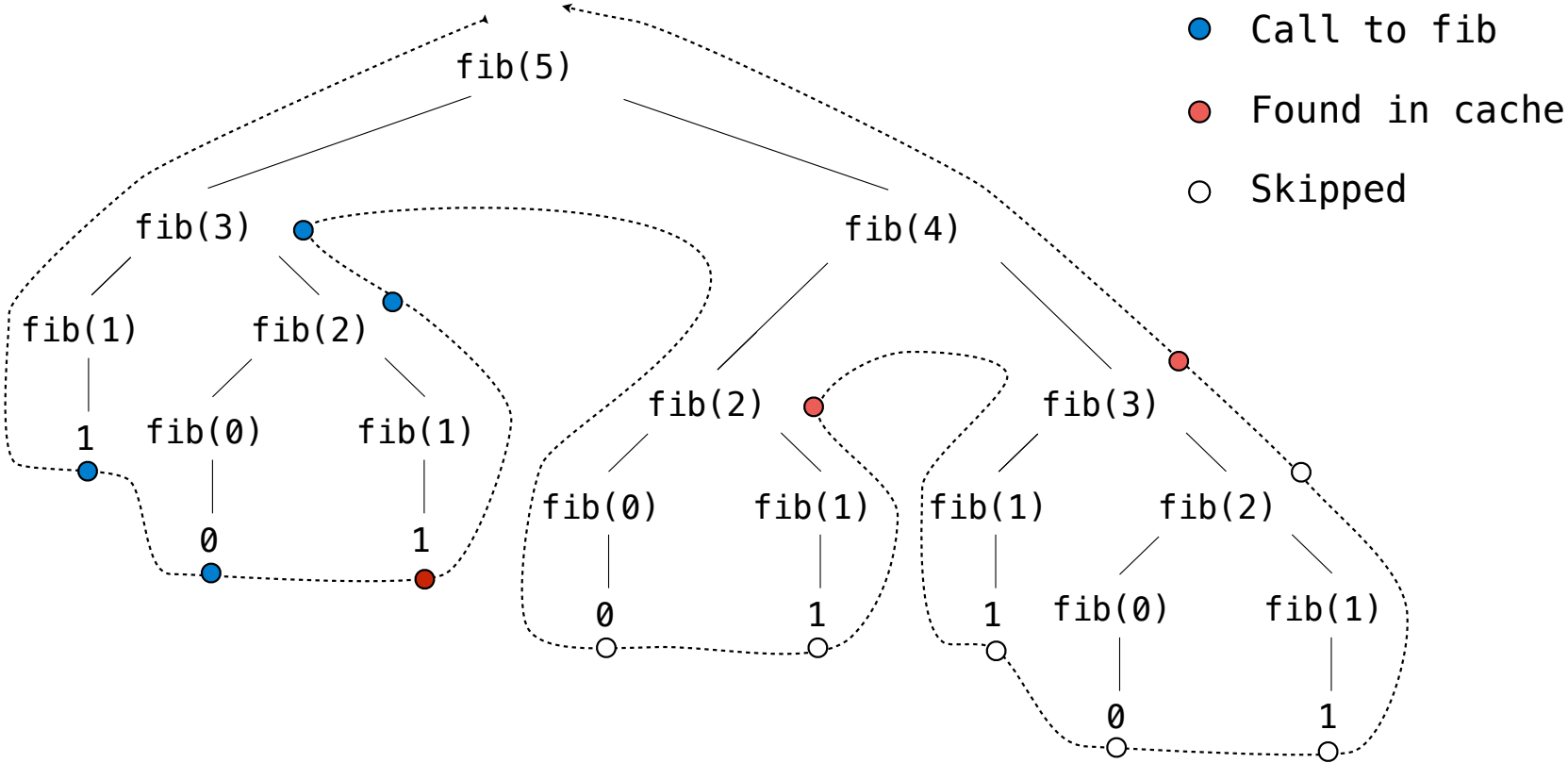
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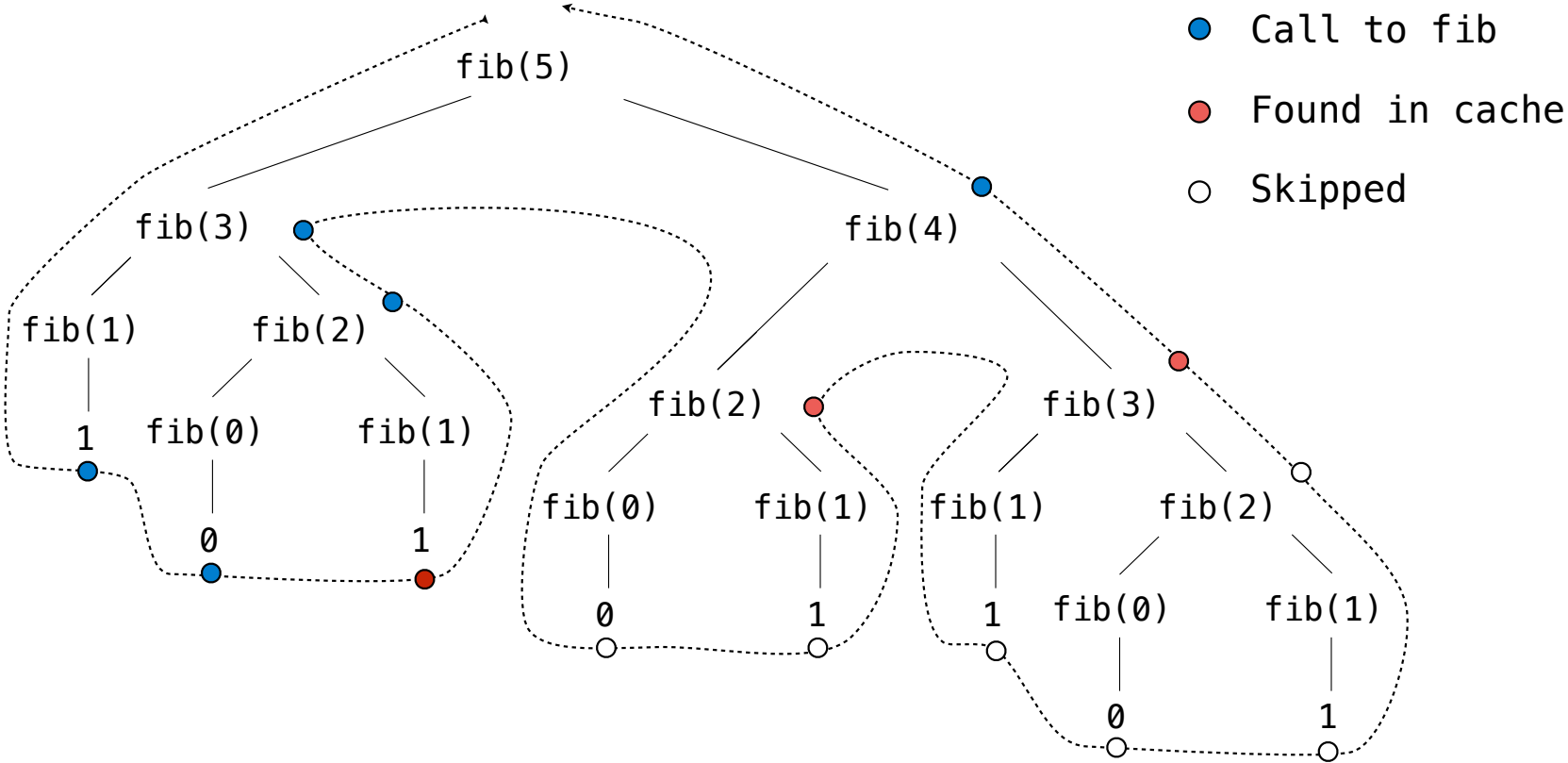
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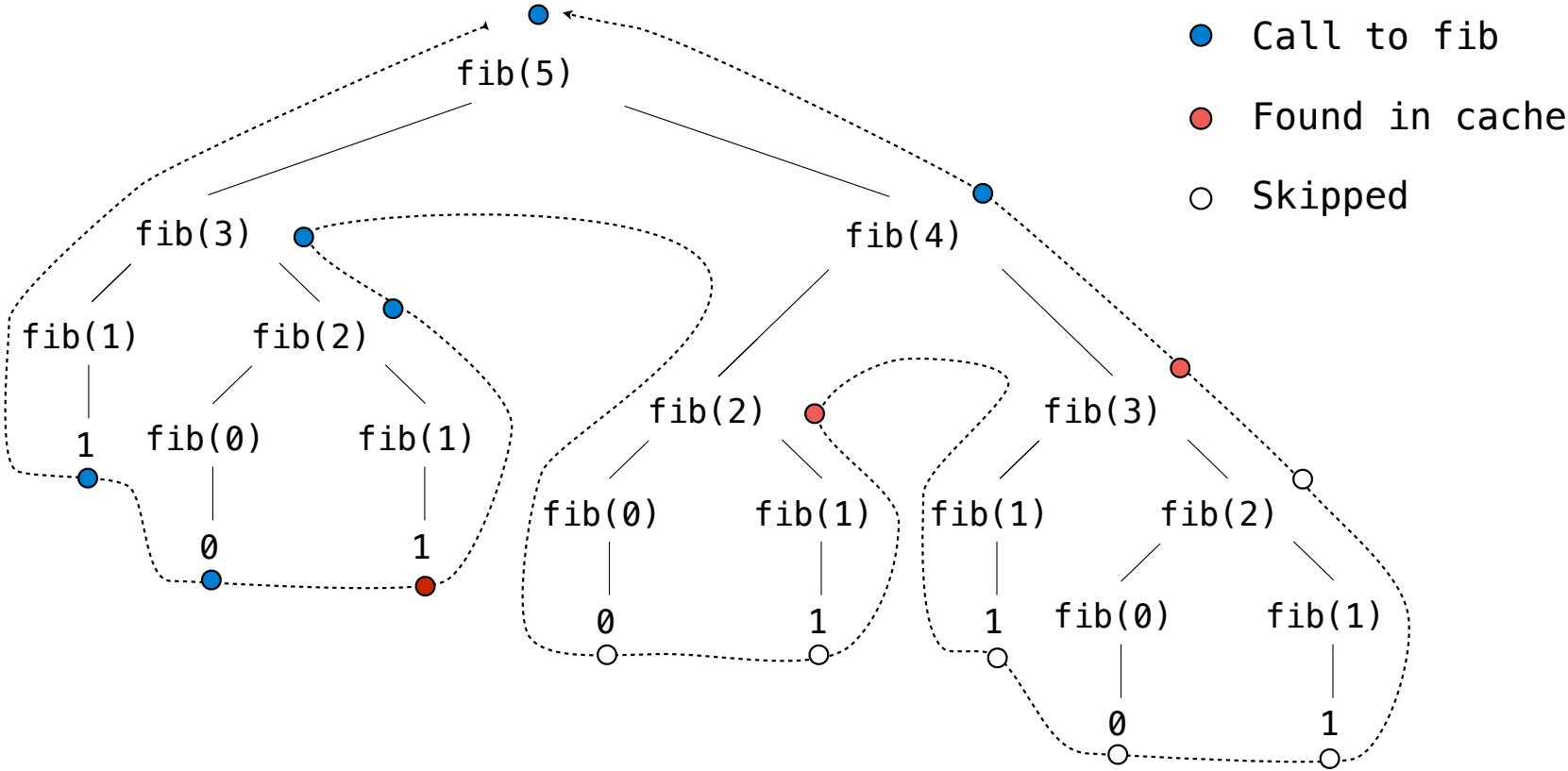
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def exp_fast(b, n):  
    if n == 0:  
        return 1  
    elif n % 2 == 0:  
        return square(exp_fast(b, n//2))  
    else:  
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Linear time:

- Doubling the input **doubles** the time
- 1024x the input takes 1024x as much time

Logarithmic time:

- Doubling the input **increases** the time by one step
- 1024x the input increases the time by only 10 steps

Orders of Growth

Quadratic Time

Functions that process all pairs of values in a sequence of length n take quadratic time

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def overlap(a, b):  
    count = 0  
    for item in a:  
        for other in b:  
            if item == other:  
                count += 1  
    return count
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overlap([3, 5, 7, 6], [4, 5, 6, 5])
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```

	3	5	7	6
4	0	0	0	0
5	0	1	0	0
6	0	0	0	1
5	0	1	0	0

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            if item == other:  
                count += 1  
    return count  
  
overlap([3, 5, 7, 6], [4, 5, 6, 5])
```

	3	5	7	6
4	0	0	0	0
5	0	1	0	0
6	0	0	0	1
5	0	1	0	0

Quadratic Time

Functions that process all pairs of values in a sequence of length n take quadratic time

```
def overlap(a, b):  
    count = 0  
    for item in a:  
        for other in b:  
            if item == other:  
                count += 1  
    return count  
  
overlap([3, 5, 7, 6], [4, 5, 6, 5])
```

	3	5	7	6
4	0	0	0	0
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(Demo)

Exponential Time

Tree-recursive functions can take exponential time

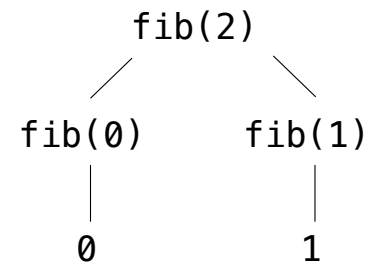
```
def fib(n):  
    if n == 0:  
        return 0  
    elif n == 1:  
        return 1  
    else:  
        return fib(n-2) + fib(n-1)
```



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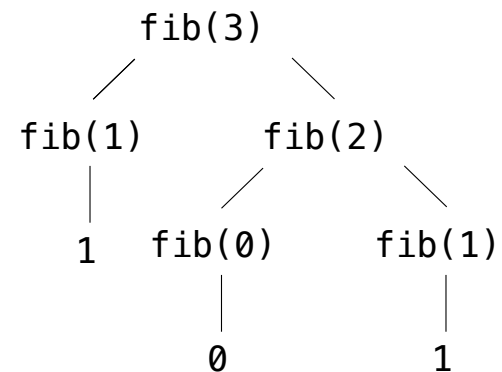
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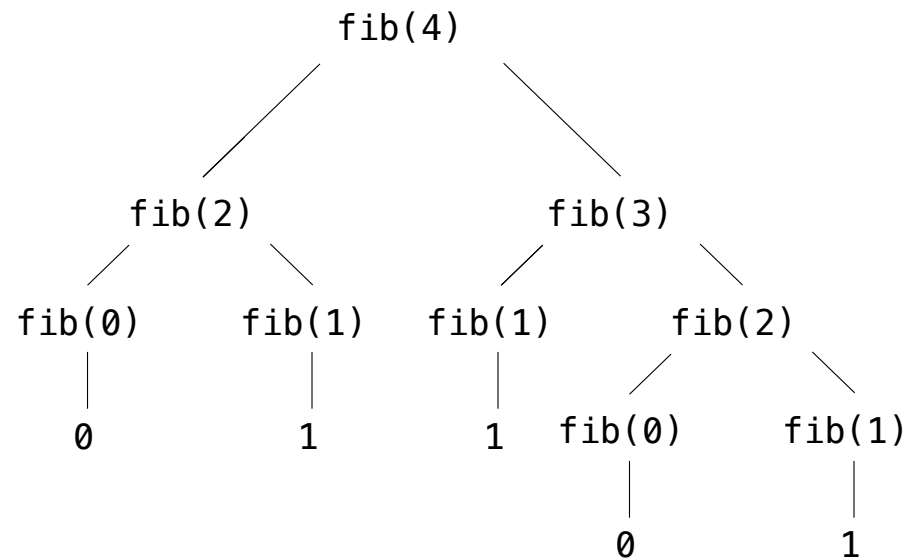
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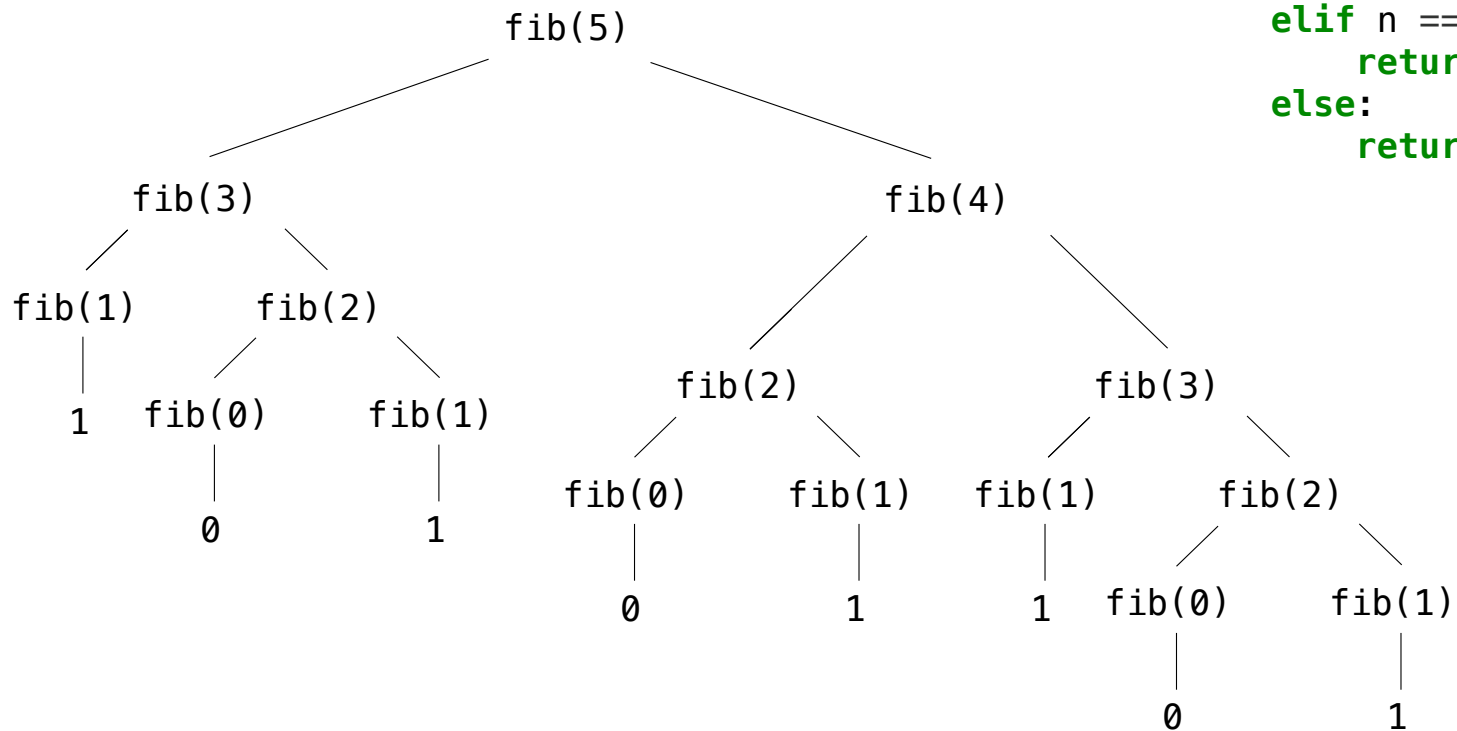
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Exponential Time

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Common Orders of Growth

Exponential growth. E.g., recursive `fib`

Quadratic growth. E.g., `overlap`

Linear growth. E.g., slow `exp`

Logarithmic growth. E.g., `exp_fast`

Constant growth. Increasing n doesn't affect time

Common Orders of Growth

Exponential growth. E.g., recursive `fib`

$$a \cdot b^{n+1} = (a \cdot b^n) \cdot b$$

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Order of Growth Notation

Big Theta and Big O Notation for Orders of Growth

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Big Theta and Big O Notation for Orders of Growth

Exponential growth. E.g., recursive `fib`

$$\Theta(b^n)$$

Incrementing n multiplies *time* by a constant

Quadratic growth. E.g., `overlap`

$$\Theta(n^2)$$

Incrementing n increases *time* by n times a constant

Linear growth. E.g., slow `exp`

$$\Theta(n)$$

Incrementing n increases *time* by a constant

Logarithmic growth. E.g., `exp_fast`

$$\Theta(\log n)$$

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Constant growth. Increasing n doesn't affect time

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Big Theta and Big O Notation for Orders of Growth

Exponential growth. E.g., recursive `fib`

Incrementing n multiplies *time* by a constant

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$$O(b^n)$$

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Space

Space and Environments

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Which environment frames do we need to keep during evaluation?

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At any moment there is a set of active environments

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Active environments:

- Environments for any function calls currently being evaluated

Space and Environments

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Active environments:

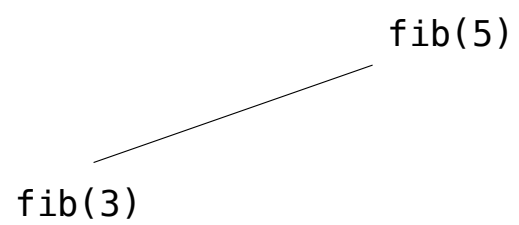
- Environments for any function calls currently being evaluated
- Parent environments of functions named in active environments

Fibonacci Space Consumption

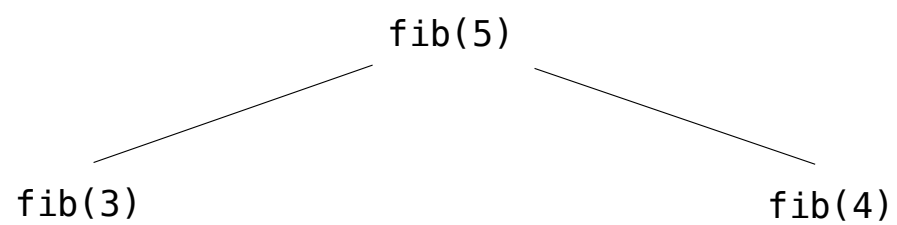
Fibonacci Space Consumption

`fib(5)`

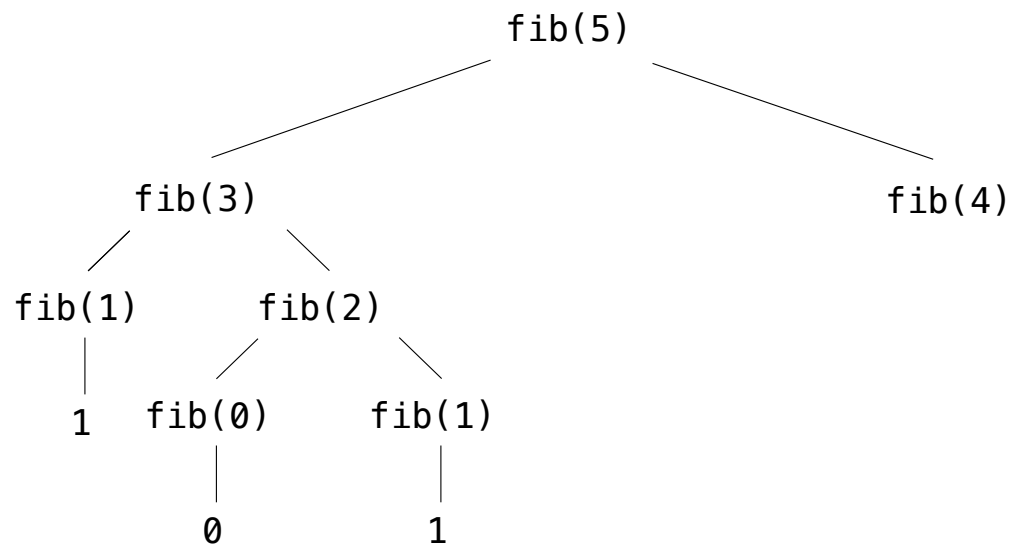
Fibonacci Space Consumption



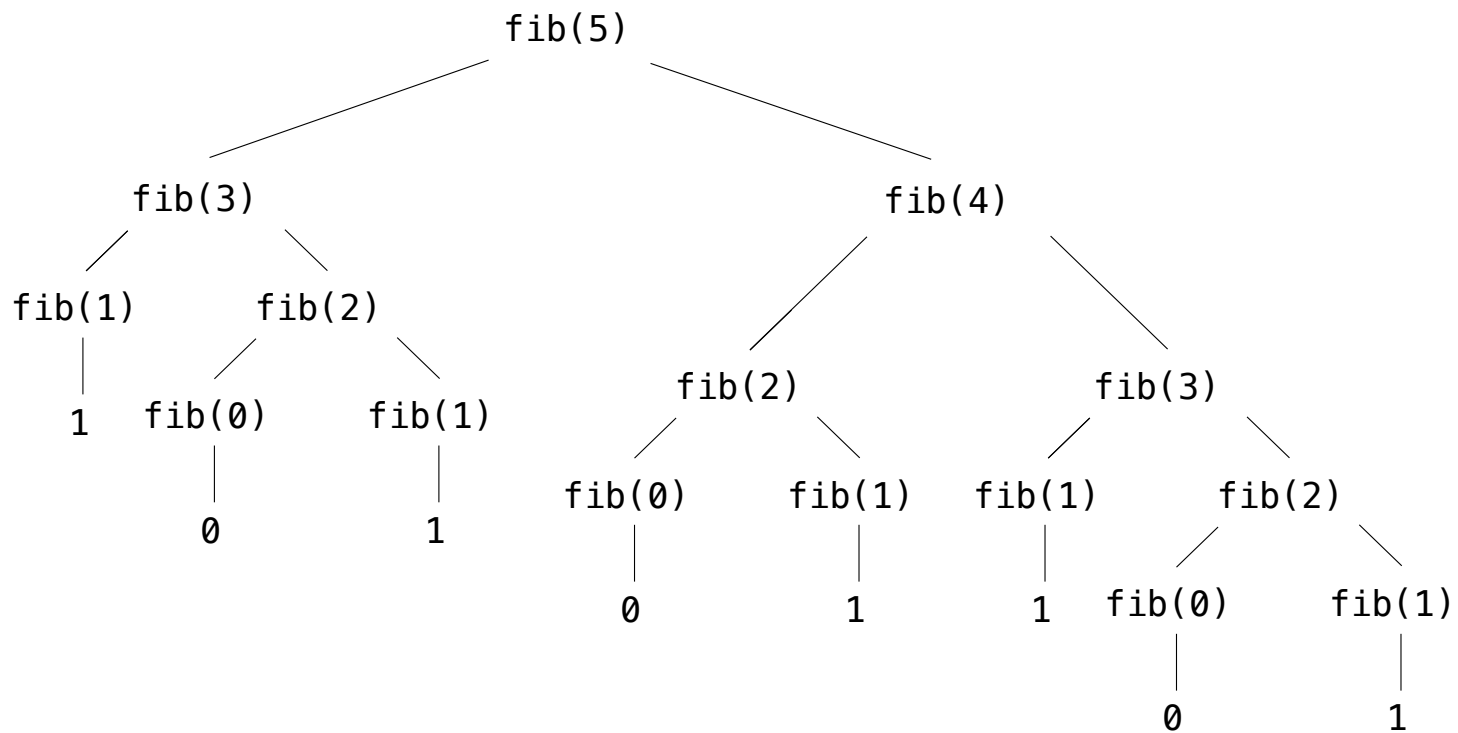
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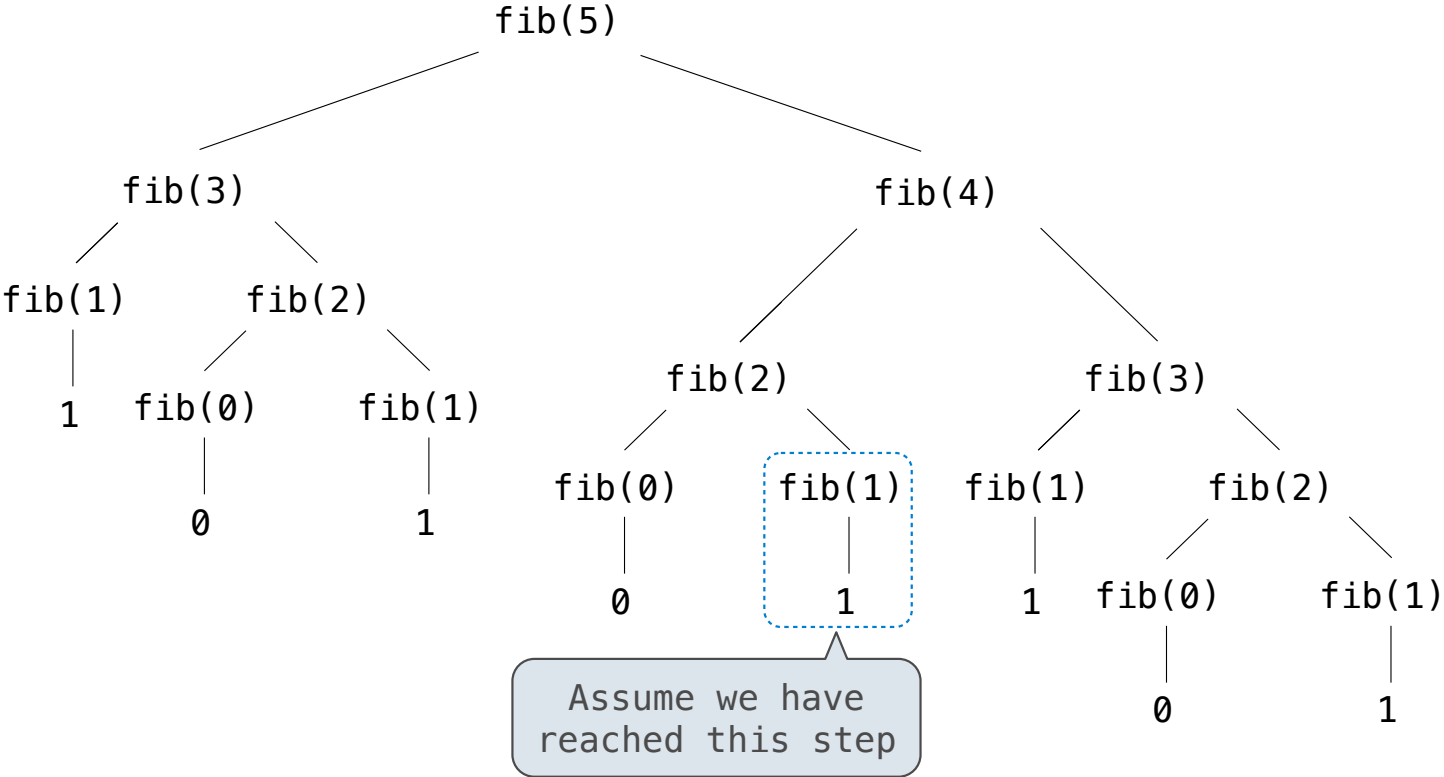
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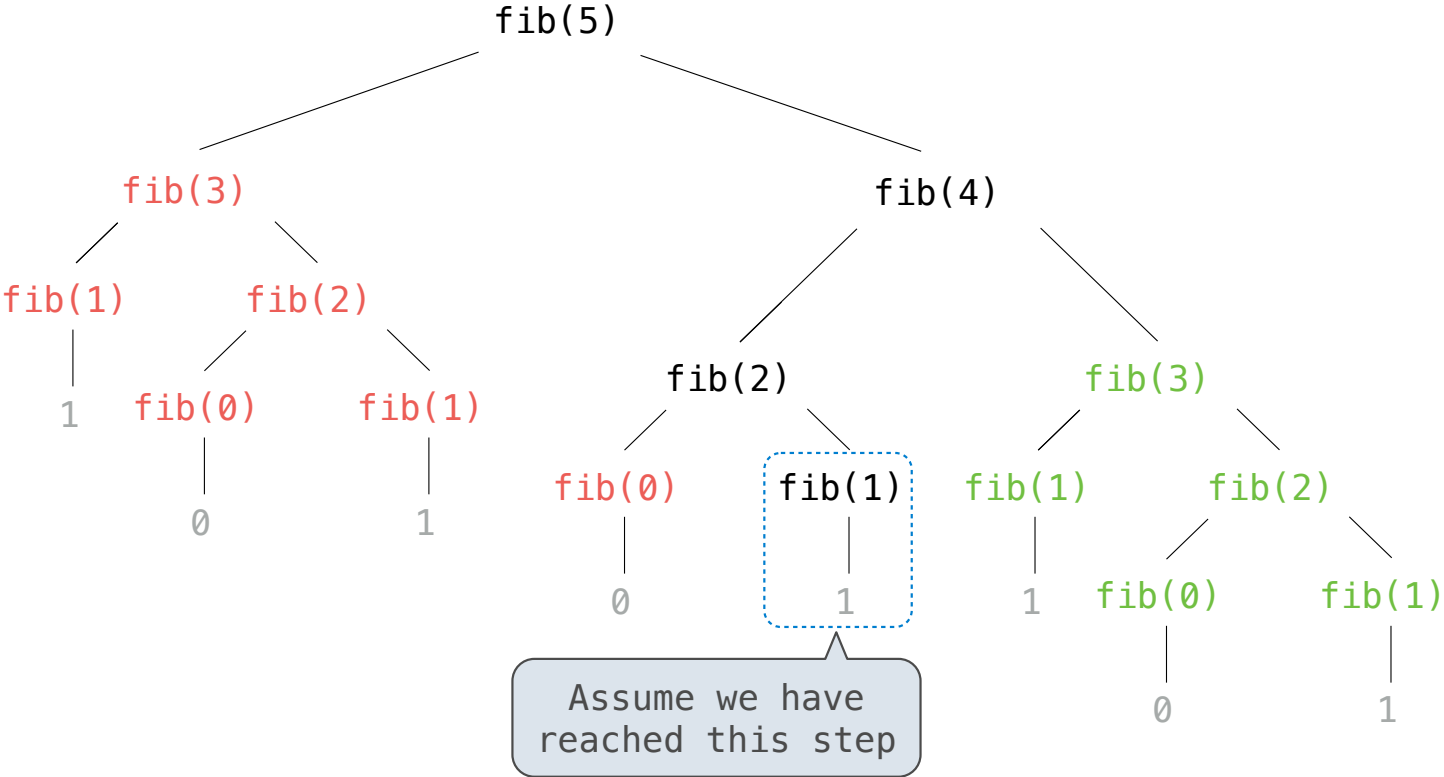
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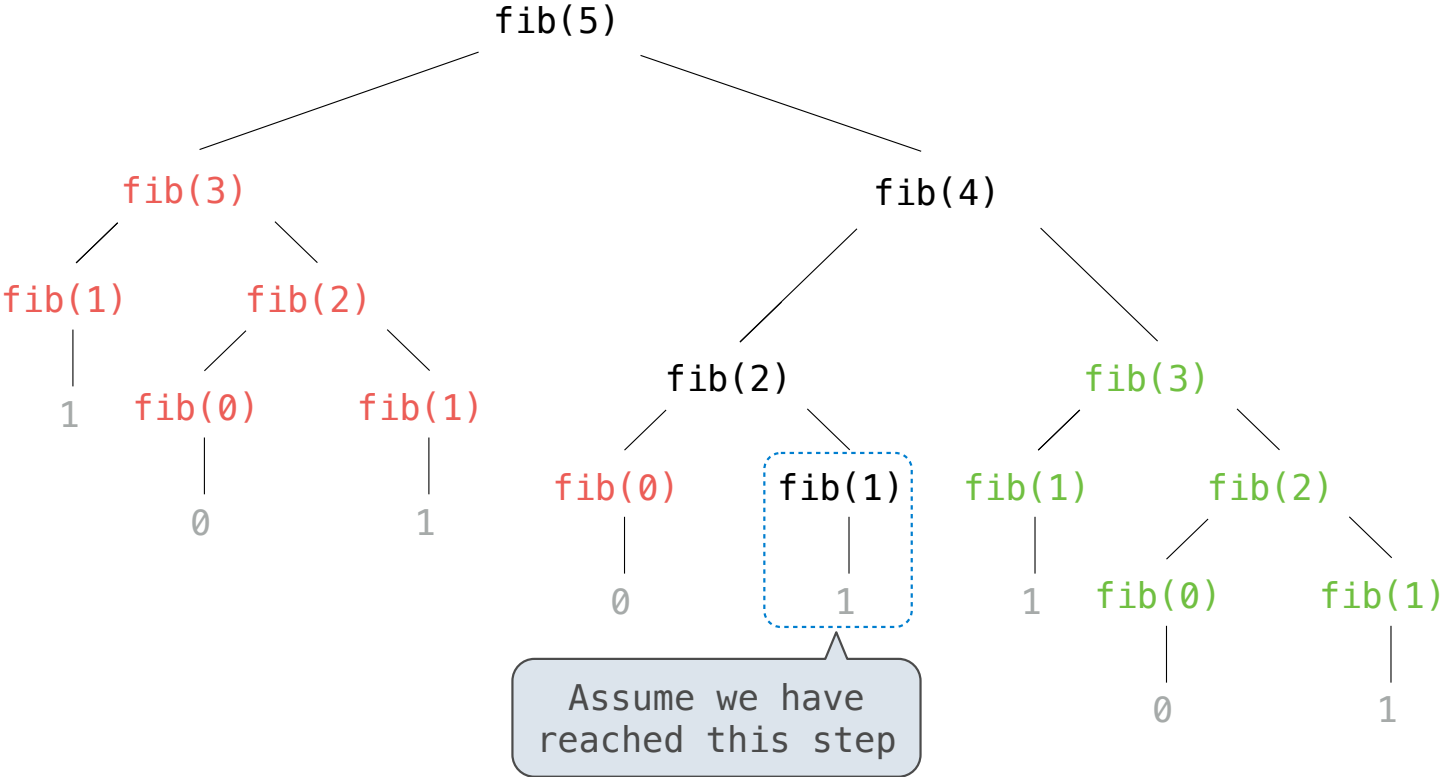


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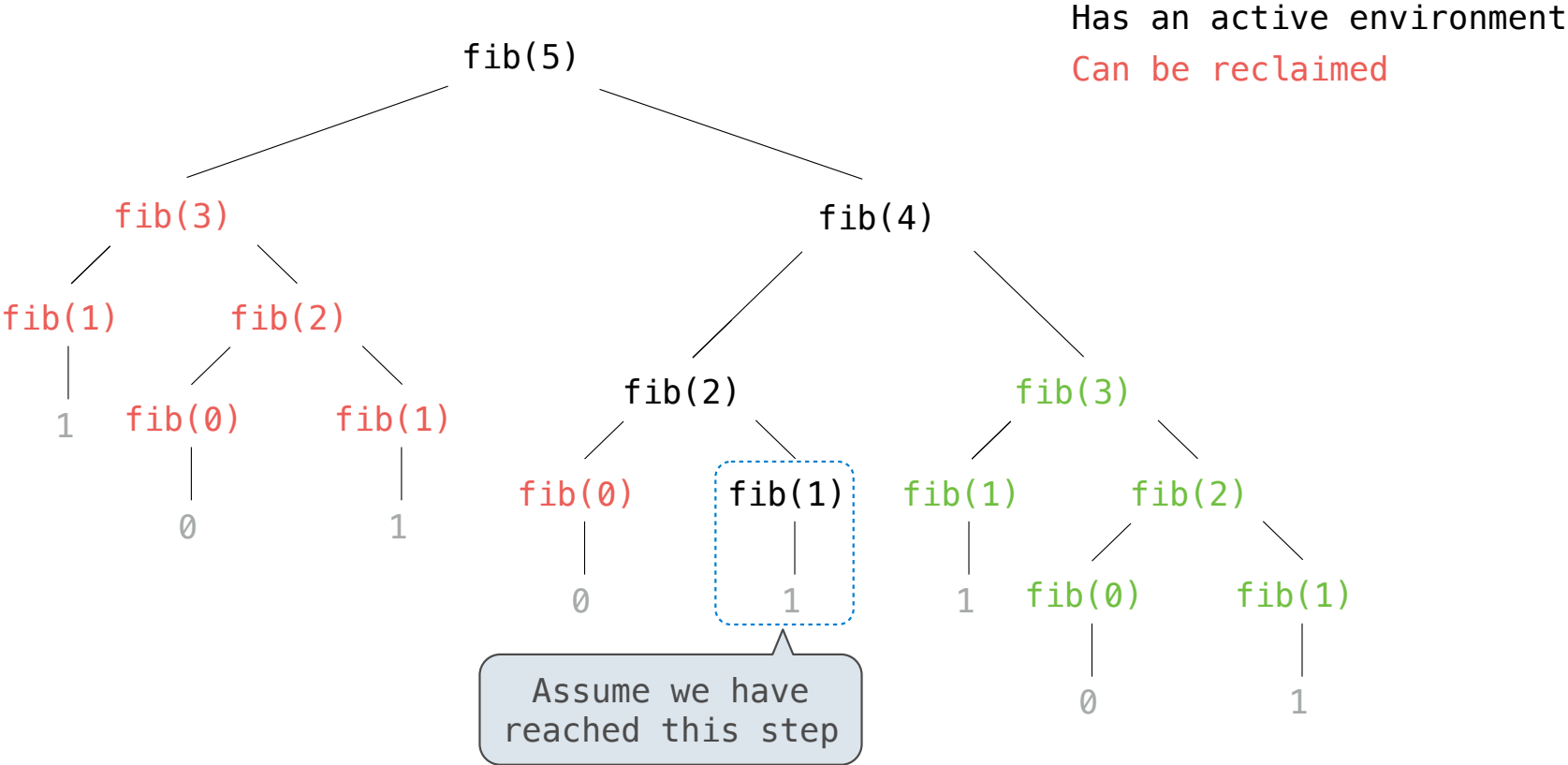


Fibonacci Space Consumption

Has an active environment



Fibonacci Space Consumption



Fibonacci Space Consumption

